## 0.1 Joining elements into structures.

### 0.1.1 Displacement and rotation field continuity

Displacement and rotation fields are continuous at the isoparametric quadrilateral inter-element interfaces; they are in fact continuous at nodes since the associated nodal Degree of Freedom (DOF)s are shared by adjacent elements, and the field interpolations that occur within each quadrilateral domain a) they both reduce to the same linear relation along the shared edge, and b) they are performed in the absence of any contributions related to unshared nodes or DOFs.

### 0.1.2 Expressing the element stiffness matrix in terms of global DOFs

As seen in Par. **??**, the stiffness matrix of each $j$-th element defines the elastic relation between the associated generalized forces and displacements, i.e.

$$\underline{\mathrm{G}}_{\,ej} = \underline{\underline{\mathrm{K}}}_{\,ej}\,\underline{\mathrm{d}}_{\,ej} \tag{1}$$

where the DOFs definition is local with respect to the element under scrutiny.

In order to investigate the mutual interaction between elements in a structure, a common set of *global* DOFs is required; in particular, generalized displacement DOFs are defined at each $l$-th global node, i.e., for nodes interacting with the shell element formulation under scrutiny,

$$\underline{\mathrm{d}}_{\,\mathrm{g}l} = \begin{bmatrix} u_{\mathrm{g}l} \\ v_{\mathrm{g}l} \\ w_{\mathrm{g}l} \\ \theta_{\mathrm{g}l} \\ \varphi_{\mathrm{g}l} \\ \psi_{\mathrm{g}l} \end{bmatrix}. \tag{2}$$

The global reference system $\mathrm{O}XYZ$ is typically employed in projecting nodal vector components. However, each $l$-th global node may be supplied with a specific reference system, whose unit vectors are $\hat{\imath}_{\mathrm{g}l}, \hat{\jmath}_{\mathrm{g}l}, \hat{k}_{\mathrm{g}l}$, thus permitting the employment of non uniformly aligned (e.g. cylindrical) global reference systems.

Those nodal degrees of freedom may be collected in a global DOFs vector

$$\underline{\mathrm{d}}_{\mathrm{g}}^{\top} = \left[\underline{\mathrm{d}}_{\mathrm{g}1}^{\top} \quad \underline{\mathrm{d}}_{\mathrm{g}2}^{\top} \quad \ldots \quad \underline{\mathrm{d}}_{\mathrm{g}l}^{\top} \quad \ldots \quad \underline{\mathrm{d}}_{\mathrm{g}n}^{\top}\right] \tag{3}$$

that parametrically defines any deformed configuration of the structure.

Analogously, a global, external (generalized[1]) forces vector may be defined, that assumes the form

$$\underline{\mathrm{F}}_{\mathrm{g}}^{\top} = \left[\underline{\mathrm{F}}_{\mathrm{g}1}^{\top} \quad \underline{\mathrm{F}}_{\mathrm{g}2}^{\top} \quad \ldots \quad \underline{\mathrm{F}}_{\mathrm{g}l}^{\top} \quad \ldots \quad \underline{\mathrm{F}}_{\mathrm{g}n}^{\top}\right]; \tag{4}$$

since *external* (single DOF or *"to ground"*) and *internal* (multi DOF) kinematic constraints are expected to be applied to the structure DOFs, the following vector of reaction forces

$$\underline{\mathrm{R}}_{\mathrm{g}}^{\top} = \left[\underline{\mathrm{R}}_{\mathrm{g}1}^{\top} \quad \underline{\mathrm{R}}_{\mathrm{g}2}^{\top} \quad \ldots \quad \underline{\mathrm{R}}_{\mathrm{g}l}^{\top} \quad \ldots \quad \underline{\mathrm{R}}_{\mathrm{g}n}^{\top}\right] \tag{5}$$

is introduced. Many FE softwares – and MSC.Marc in particular – treat external and internal constraints separately, thus leading to two set of constraint actions, namely the (strictly named) *reaction forces*, and the *tying forces*, respectively; for the sake of simplicity, the constraint treatise is unified in the present contribution.

The simple four element, roof-like structure of Fig. 1 is employed in the following to discuss the procedure that derives the elastic response characterization for the structure from its elemental counterparts.

The structure comprises nine nodes, whose location in space is defined according to a global reference system $\mathrm{O}XYZ$, see Table 1.

The structure is composed by four, identical, four noded isoparametric shell elements, whose formulation is described in the preceding section **??**.

A grayscale, normalized representation of the element stiffness matrix is shown in Figure 2, where the white to black colormap spans from zero to the maximum in absolute value term.

The mapping between local, element based node numbering and the global node numbering is reported in the connectivity Table 2.

Such i) local to global node numbering mapping, together with ii) the change in reference system mentioned above, defines a set of

[1]Unless otherwise specified, the *displacement* and *force* terms refer to the DOFs, and the suitable actions that perform work with their variation, respectively. They are in fact *generalized* forces and displacements.

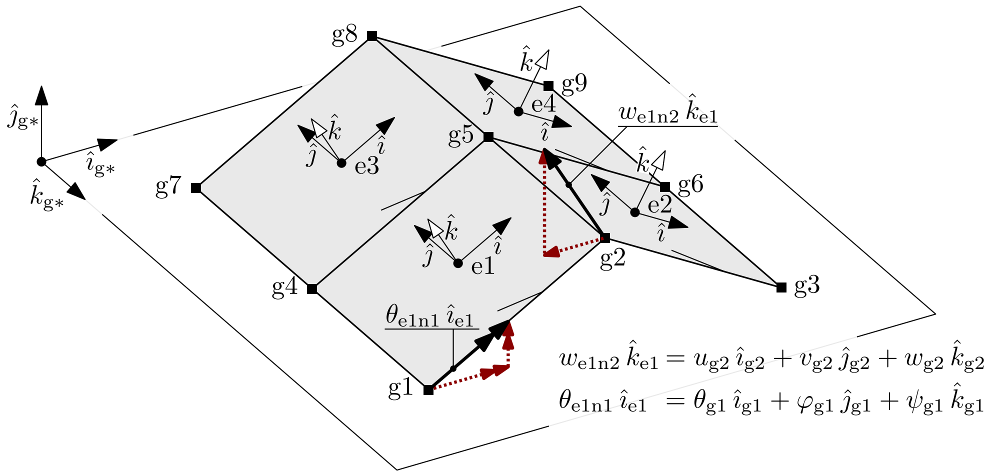



Figure 1: A simple four-element, roof-like structure employed in discussing the assembly procedures. The elements are square, thick plates whose angle with respect to the global $XY$ plane is 30°

| node | $X$ | $Y$ | $Z$ |
|---|---|---|---|
| g1 | $-lc$ | 0 | $+l$ |
| g2 | 0 | $+ls$ | $+l$ |
| g3 | $+lc$ | 0 | $+l$ |
| g4 | $-lc$ | 0 | 0 |
| g5 | 0 | $+ls$ | 0 |
| g6 | $+lc$ | 0 | 0 |
| g7 | $-lc$ | 0 | $-l$ |
| g8 | 0 | $+ls$ | $-l$ |
| g9 | $+lc$ | 0 | $-l$ |

Table 1: Nodal coordinates for the roof-like structure of Figure 1. $l$ is the element side length, $c = \cos 30°$ and $s = \sin 30°$

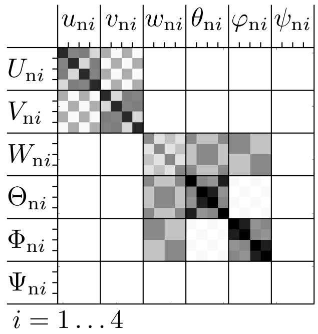



Figure 2: A representation of the stiffness matrix terms for each element in the example structure; the term magnitude is represented through a linear grayscale, spanning from zero (white) to the peak value (black).

| | n1 | n2 | n3 | n4 |
|---|---|---|---|---|
| e1 | g1 | g2 | g5 | g4 |
| e2 | g2 | g3 | g6 | g5 |
| e3 | g4 | g5 | g8 | g7 |
| e4 | g5 | g6 | g9 | g8 |

Table 2: Element connectivity for the roof-like structure of Figure 1. As an example, the node described by the local numbering e3n2 is mapped to the global node g5.

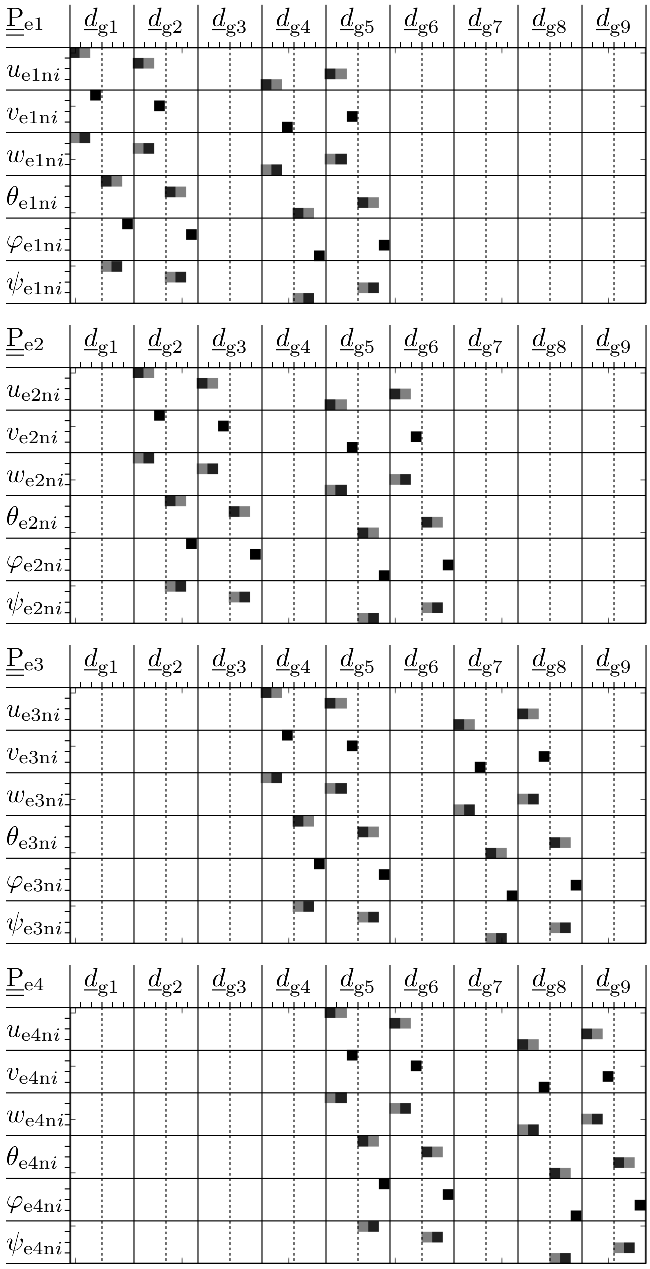


Figure 3: A grayscale representation of the terms of the four $\underline{\underline{\mathrm{P}}}_{\mathrm{e}j}$ mapping matrices associated the elements of Fig. 1. The colormap spans from white (zero) to black (one); the lighter and the darker grey colors represent terms that equate in modulus $\sin 30°$ and $\cos 30°$, respectively.

elemental DOF mapping matrices, $\underline{\underline{\mathrm{P}}}_{\mathrm{e}j}$, one each $j$-th element. Such matrices are defined as follows: the $i$-th row the $\underline{\underline{\mathrm{P}}}_{\mathrm{e}j}$ matrix contains the coefficients of the linear combination of global DOFs that equates the $i$ local DOF of the $j$-th element; an example is proposed in the following to illustrate such relation.

With reference to the structure of Figure 1, $w_{\mathrm{e1n2}}$ and $\theta_{\mathrm{e1n1}}$ respectively represent the 10th and the 13th local degrees of freedom of element 1.

Their global representation involves a subset of the g2 and g1 global nodes DOFs, respectively, namely

$$w_{\mathrm{e1n2}} = \langle \hat{k}_{\mathrm{e1}}, \hat{\imath}_{\mathrm{g2}} \rangle u_{\mathrm{g2}} + \langle \hat{k}_{\mathrm{e1}}, \hat{\jmath}_{\mathrm{g2}} \rangle v_{\mathrm{g2}} + \langle \hat{k}_{\mathrm{e1}}, \hat{k}_{\mathrm{g2}} \rangle w_{\mathrm{g2}} \tag{6}$$

$$\theta_{\mathrm{e1n1}} = \langle \hat{\imath}_{\mathrm{e1}}, \hat{\imath}_{\mathrm{g1}} \rangle \theta_{\mathrm{g1}} + \langle \hat{\imath}_{\mathrm{e1}}, \hat{\jmath}_{\mathrm{g1}} \rangle \phi_{\mathrm{g1}} + \langle \hat{\imath}_{\mathrm{e1}}, \hat{k}_{\mathrm{g1}} \rangle \psi_{\mathrm{g1}} \tag{7}$$

where $\hat{\imath}_{\mathrm{e1}}$,$\hat{\jmath}_{\mathrm{e1}}$, $\hat{k}_{\mathrm{e1}}$ are the orientation vectors of the element 1 local reference system, $\hat{\imath}_{\mathrm{g1}}$,$\hat{\jmath}_{\mathrm{g1}}$,$\hat{k}_{\mathrm{g1}}$ and $\hat{\imath}_{\mathrm{g2}}$,$\hat{\jmath}_{\mathrm{g2}}$,$\hat{k}_{\mathrm{g2}}$ are the orientation vectors of the global nodes 1 and 2 reference systems, and $\langle \cdot, \cdot \rangle$ represents their mutual scalar product, or, equivalently, the cosinus of the angle between two unit vectors.

The 10th and the 13th row of the $\underline{\underline{\mathrm{P}}}_{\mathrm{e1}}$ mapping matrix are defined based on Eqs.6 and 7, respectively, and they are null except for the elements

$$\begin{aligned}
\left[\underline{\underline{\mathrm{P}}}_{\mathrm{e1}}\right]_{10,7} &= \langle \hat{k}_{\mathrm{e1}}, \hat{\imath}_{\mathrm{g2}} \rangle & \left[\underline{\underline{\mathrm{P}}}_{\mathrm{e1}}\right]_{13,4} &= \langle \hat{\imath}_{\mathrm{e1}}, \hat{\imath}_{\mathrm{g1}} \rangle \\
\left[\underline{\underline{\mathrm{P}}}_{\mathrm{e1}}\right]_{10,8} &= \langle \hat{k}_{\mathrm{e1}}, \hat{\jmath}_{\mathrm{g2}} \rangle & \left[\underline{\underline{\mathrm{P}}}_{\mathrm{e1}}\right]_{13,5} &= \langle \hat{\imath}_{\mathrm{e1}}, \hat{\jmath}_{\mathrm{g1}} \rangle \\
\left[\underline{\underline{\mathrm{P}}}_{\mathrm{e1}}\right]_{10,9} &= \langle \hat{k}_{\mathrm{e1}}, \hat{k}_{\mathrm{g2}} \rangle & \left[\underline{\underline{\mathrm{P}}}_{\mathrm{e1}}\right]_{13,6} &= \langle \hat{\imath}_{\mathrm{e1}}, \hat{k}_{\mathrm{g1}} \rangle,
\end{aligned}$$

being $u_{\mathrm{g2}}$,$v_{\mathrm{g2}}$,$w_{\mathrm{g2}}$,$\theta_{\mathrm{g1}}$,$\phi_{\mathrm{g1}}$ and $\psi_{\mathrm{g1}}$ the 7th, 8th, 9th, 4th, 5th and 6th global degrees of freedom according to their position in $\underline{\mathrm{d}}_{\mathrm{g}}$.

Figure 3 presents a grayscale representation of the four $\underline{\underline{\mathrm{P}}}_{\mathrm{e}j}$ matrices; please note the extremely sparse nature of those matrices, whose number of nonzero terms scales with the single element DOF cardinality, whereas the total number of terms scale with the whole structure DOF cardinality.

The rows of the rectangular $\underline{\underline{\mathrm{P}}}_{\mathrm{e}j}$ mapping matrix are mutually orthonormal; the mapping matrix is orthogonal in the sense of the Moore-Penrose pseudoinverse, since its transpose and its pseudoinverse coincide.

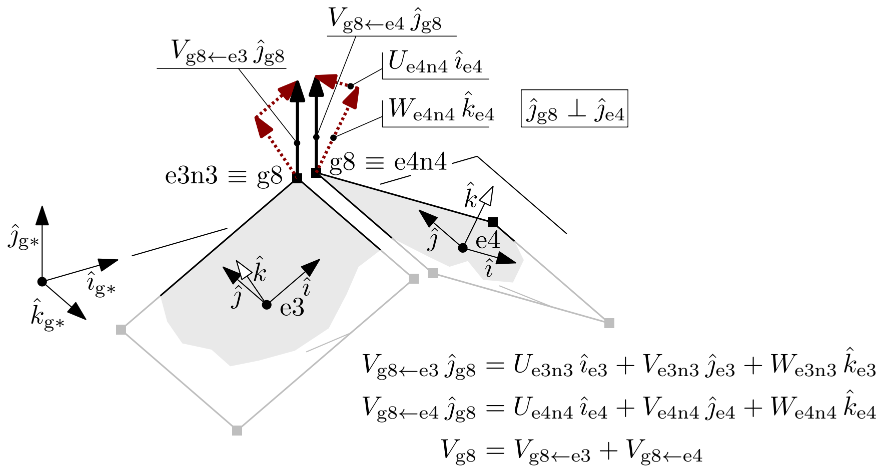



Figure 4: Accumulation of elemental nodal actions at global nodes.

The elemental mapping $\underline{\underline{\mathrm{P}}}_{\mathrm{e}j}$ matrices constitute an artifice that plays a double role in the local to global DOF mapping; if on one side the $j$-th element DOFs may be derived from their global counterpart as

$$\underline{\mathrm{d}}_{\mathrm{e}j} = \underline{\underline{\mathrm{P}}}_{\mathrm{e}j}\,\underline{\mathrm{d}}_{\mathrm{g}}, \quad \forall j \tag{8}$$

on the other, the nodal actions required to oppose the elastic reactions at each $j$-th element, as evaluated as in Eq. 1 according to the local DOF system, may be collected at global nodes.

Such collection is illustrated in Figure 4 for the second DOF of the global node g8, and in particular the force component $V_{\mathrm{g8}}$, namely the 44-th component of $\underline{\mathrm{G}}_g$; such a force component collects the contributions aligned with the $\hat{\jmath}_{\mathrm{g8}}$ unit vector from element 3, local node 3, and element 4, local node 4, named $V_{\mathrm{g8}\leftarrow\mathrm{e3}}$ and $V_{\mathrm{g8}\leftarrow\mathrm{e4}}$, respectively.

Figure 4 equations relate the nodal force components expressed with respect to the element reference systems with the global force component under scrutiny; in particular we have

$$V_{\mathrm{g8}\leftarrow\mathrm{e3}} = U_{\mathrm{e3n3}}\langle\hat{\imath}_{\mathrm{e3}}, \hat{\jmath}_{\mathrm{g8}}\rangle + V_{\mathrm{e3n3}}\langle\hat{\jmath}_{\mathrm{e3}}, \hat{\jmath}_{\mathrm{g8}}\rangle + W_{\mathrm{e3n3}}\langle\hat{k}_{\mathrm{e3}}, \hat{\jmath}_{\mathrm{g8}}\rangle \tag{9}$$

$$V_{\mathrm{g8}\leftarrow\mathrm{e4}} = U_{\mathrm{e4n4}}\langle\hat{\imath}_{\mathrm{e4}}, \hat{\jmath}_{\mathrm{g8}}\rangle + V_{\mathrm{e4n4}}\langle\hat{\jmath}_{\mathrm{e4}}, \hat{\jmath}_{\mathrm{g8}}\rangle + W_{\mathrm{e4n4}}\langle\hat{k}_{\mathrm{e4}}, \hat{\jmath}_{\mathrm{g8}}\rangle. \tag{10}$$

If we want to collect the contribution along the global DOFs of the forces collected on element 4 in the algebraic relation

$$\underline{\mathrm{G}}_{g\leftarrow \mathrm{e}4} = \underline{\underline{\mathrm{P}}}'_{\mathrm{e}4}\,\underline{\mathrm{G}}_{\mathrm{e}4} \tag{11}$$

the 44-th row of the $\underline{\underline{\mathrm{P}}}'$ - whose row and column cardinality equates that of the global and the elemental DOF, respectively - may be compiled based on 10; in particular, its nonzero terms are

$$\left[\underline{\underline{\mathrm{P}}}'_{\mathrm{e}4}\right]_{44,4} = \langle \hat{\jmath}_{\mathrm{g}8}, \hat{\imath}_{\mathrm{e}4}\rangle \qquad \left[\underline{\underline{\mathrm{P}}}'_{\mathrm{e}4}\right]_{44,12} = \langle \hat{\jmath}_{\mathrm{g}8}, \hat{k}_{\mathrm{e}4}\rangle$$
$$\left[\underline{\underline{\mathrm{P}}}'_{\mathrm{e}4}\right]_{44,8} = \langle \hat{\jmath}_{\mathrm{g}8}, \hat{\jmath}_{\mathrm{e}4}\rangle$$

being 4,8,12 the index locations of $U_{\mathrm{e4n4}}, V_{\mathrm{e4n4}}, W_{\mathrm{e4n4}}$ within $\underline{\mathrm{G}}_{\mathrm{e}4}$.

By repeating the procedure for each global DOF, and for each element, it is found that the $\underline{\underline{\mathrm{P}}}'_{ej}$ matrices equate the transpose of the $\underline{\underline{\mathrm{P}}}_{\mathrm{e}j}$ matrices associated to the same element, and hence Eq. 10 may be recast for each element as

$$\underline{\mathrm{G}}_{g\leftarrow \mathrm{e}j} = \underline{\underline{\mathrm{P}}}^{\top}_{\mathrm{e}j}\,\underline{\mathrm{G}}_{\mathrm{e}j}, \quad \forall j \tag{12}$$

thus obtaining a transformation from element DOFs to their contributes to global counterparts.

The role of $\underline{\underline{\mathrm{P}}}^{\top}_{ej}$ in such a local-to-global mapping Eq. pairs the role of $\underline{\underline{\mathrm{P}}}_{\mathrm{e}j}$ in the global-to-local relation expressed in Eq. . A strict inverse relation may not be defined due to the different cardinality of the two DOF sets, and $\underline{\underline{\mathrm{P}}}_{\mathrm{e}j}$ lacks of a proper inverse, being in fact a rectangular matrix.

However, due to the mutually orthonormal nature of the $\underline{\underline{\mathrm{P}}}_{\mathrm{e}j}$ matrix columns, such a matrix may be defined *orthonormal* in the sense of the Moore-Penrose pseudoinverse; the $\underline{\underline{\mathrm{P}}}^{\top}_{\mathrm{e}j}$ matrixes that, for each element, control the local-to-global mapping are the pseudoinverses of the $\underline{\underline{\mathrm{P}}}_{\mathrm{e}j}$ matrixes that regulate the global-to-local mapping.

Based on 1, 8 and 12, the contribution of the $j$-th element to the elastic response of the structure may finally be described as the vector of global force components

$$\underline{\mathrm{G}}_{\mathrm{g}\leftarrow \mathrm{e}j} = \underline{\underline{\mathrm{P}}}^{\top}_{\mathrm{e}j}\,\underline{\underline{\mathrm{K}}}_{\mathrm{e}j}\,\underline{\underline{\mathrm{P}}}_{\mathrm{e}j}\,\underline{\mathrm{d}}_{\mathrm{g}}; \tag{13}$$

that have to be applied at the structure DOFs in order to equilibrate the elastic reactions that arise at the nodes of the $j$-th element, if a deformed configuration is prescribed for the latter according to the $\underline{\mathrm{d}}_{\mathrm{g}}$ global displacement mode.

By accumulating the contribution of the various elements in a structure, the overall relation is obtained

$$\underline{\mathrm{G}}_{\mathrm{g}} = \sum_j \underline{\mathrm{G}}_{\mathrm{g}\leftarrow \mathrm{e}j} = \left( \sum_j \underline{\underline{\mathrm{P}}}_{\mathrm{e}j}^{\top} \underline{\underline{\mathrm{K}}}_{\mathrm{e}j} \underline{\underline{\mathrm{P}}}_{\mathrm{e}j} \right) \underline{\mathrm{d}}_{\mathrm{g}} = \underline{\underline{\mathrm{K}}}_{\mathrm{g}} \underline{\mathrm{d}}_{\mathrm{g}}, \tag{14}$$

that defines the $\underline{\underline{\mathrm{K}}}_{\mathrm{g}}$ global stiffness matrix as an assembly of the elemental contributions. The contribute accumulation at each summatory step is graphically represented in Fig. 5, in the case of the example structure of Fig. 1.

The global stiffness matrix is symmetric, and it shows nonzero terms at cells whose row and column indices are associate to two DOFs that are bridged by a direct elastic link – i.e., an element exists, that insists on both the nodes those DOFs pertain; since only a limited number of elements insist on each given node, the matrix is sparse, as shown in Fig. 5d.

An favourable numbering of the global nodes may be searched for, such that the nonzero terms are clustered within a (possibly) narrow band around the diagonal; the resulting stiffness matrix is hence *banded*, condition this that reduces both the storage memory requirements, and the computational effort in applying the various algebraic operators to the matrix.

The stiffness matrix (half-)bandwidth may be predicted by evaluating the bandwidth required for storing each element contribution

$$b_{\mathrm{e}j} = (i_{\max} - i_{\min} + 1)\, l, \tag{15}$$

and retaining the

$$b = \max_{\mathrm{e}j} b_{\mathrm{e}j} \tag{16}$$

peak value; in the formula 15, $l$ is the number of DOF per element node, whereas $i_{\max}$ and $i_{\max}$ are the extremal integer labels associated to the element nodes, according to the global numbering.

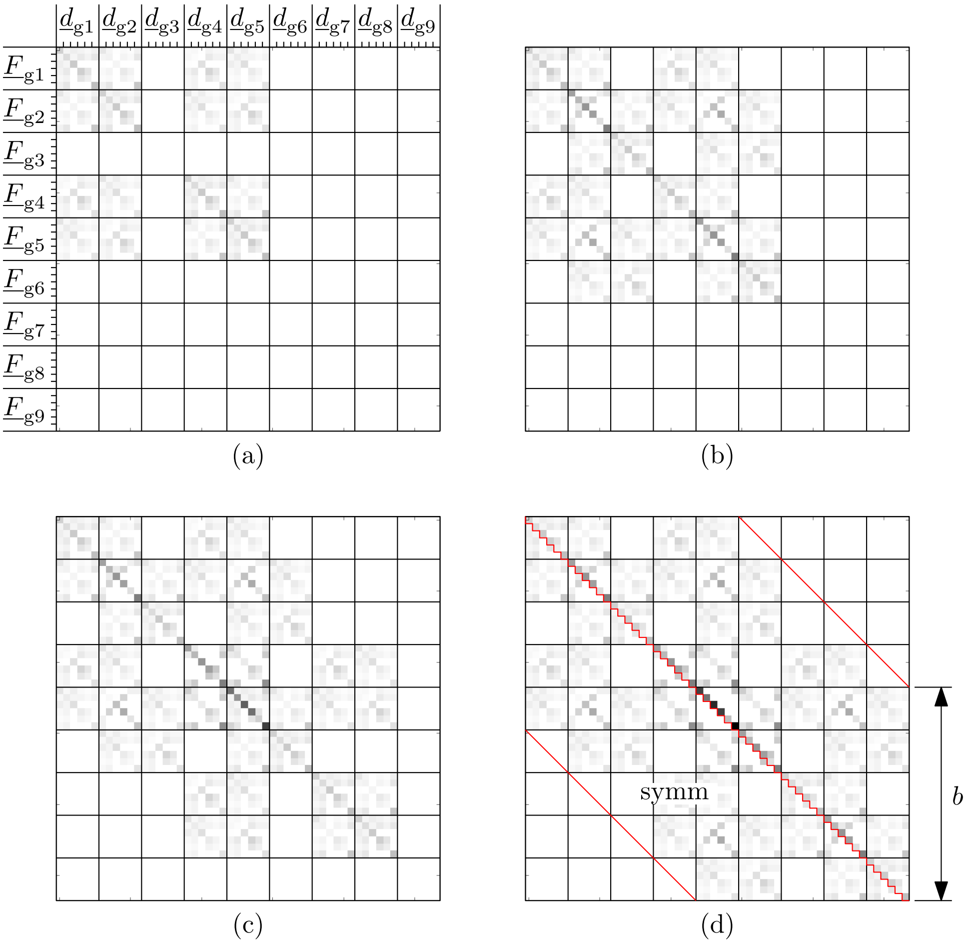



Figure 5: Graphical representation of the assembly steps for the stiffness matrix of the Fig. 1 structure. The zero-initialized form for the matrix that precedes the (a) step is omitted.

### 0.1.3 External forces assembly

The element vector forces are accumulated to derive global external forces vector $\underline{\mathrm{F}}_{\mathrm{g}}$, as in

$$\underline{\mathrm{F}}_{\mathrm{g}} = \sum_j \underline{\underline{\mathrm{P}}}_{\mathrm{e}j}^{\top} \underline{\mathrm{F}}_{\mathrm{e}j}; \tag{17}$$

the transposed $\underline{\underline{\mathrm{P}}}_{\mathrm{e}j}^{\top}$ mapping matrix is employed to translate the actions on the local DOFs to their global counterpart.

## 0.2 Constraints.

### 0.2.1 A pedagogical example.

Figure 6 represents a simple, pedagogical example of a three d.o.f. elastic system subject to a set of two kinematic constraints. The first, I, embodies a typical multi d.o.f. constraint[2], namely a 3:1 leverage between the vertical displacements $d_3$ and $d_1$ The second, II, consists in a single d.o.f., inhomogeneous constraint that imposes a fixed value to the $d_2$ vertical displacement.

Both the kinematic constraint may be cast in the same algebraic form

$$\sum_i \alpha_{ji} \underline{\mathrm{d}}_i = \underline{\alpha}_j^\top \underline{\mathrm{d}} = \Delta_j \tag{18}$$

where $j = I, II$ and $i = 1 \ldots 3$ the indexes span through the constraints and the model d.o.f.s, respectively, and the $\underline{\alpha}_j$ equation coefficient vectors and inhomogeneous terms are

$$\begin{aligned} \underline{\alpha}_{\mathrm{I}}^\top &= \begin{bmatrix} 3 & 0 & 1 \end{bmatrix} & \Delta_{\mathrm{I}} &= 0 \\ \underline{\alpha}_{\mathrm{II}}^\top &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} & \Delta_{\mathrm{II}} &= 0.2 \end{aligned}$$

In the absence of constraints, viable system configurations span the whole $\mathbb{R}^3$ space of Fig. 7 (a); viable configurations with respect to the first constraint alone span the *hyper*-plane/subspace[3] I, whereas the subspace II collects the feasible configurations with respect to the second constraint.

It is relevant to underline that the feasible configuration hyperplanes I and II are normal to the associated coefficient vectors $\underline{\alpha}_{\mathrm{I}}$ and $\underline{\alpha}_{\mathrm{II}}$, respectively.

The $\mathrm{I} \cap \mathrm{II}$ intersection subspace collects the configurations that satisfies both the constraints; such subspace is orthogonal to both $\underline{\alpha}_{\mathrm{I}}$ and $\underline{\alpha}_{\mathrm{II}}$.

If the constraints are assumed as ideal[4], the exerted reactions are orthogonal to the allowed displacements; reaction forces are confined

[2]usually, and rather improperly, named *multipoint* constraint (MPC)

[3]The subspace of the feasible configurations with respect to a single, scalar linear equation is an hyperplane in the configuration space; due to the limited d.o.f. set cardinality, Figure 7 (a) represents a 2d plane within a 3d space. The *hyper*-nomenclature is preserved to

[4]or, namely, frictionless

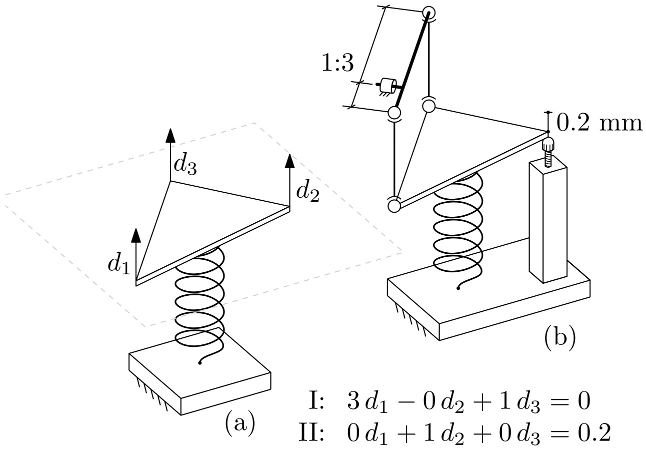



Figure 6: A pedagogical elastic three d.o.f. system, (a), subject to a few kinematic constraints (b).

on a subspace of the reaction space that corresponds to[5] the orthogonal complement of the feasible subspace of the configuration space.

By moving on the constraint reaction space shown in 7 (b), the reaction forces associated to constraint I and II are thus proportional to the $\underline{\alpha}_{\mathrm{I}}$ and $\underline{\alpha}_{\mathrm{II}}$ vectors, respectively; the cumulative constraint reactions lie on the linear span of those two vectors, namely $\mathcal{L}\left(\alpha_{\mathrm{I}}, \alpha_{\mathrm{II}}\right)$.

With reference to some concepts anticipated from the next paragraph, we may set $d_1$ as the only retained[6] DOF, thus leading to $\underline{\underline{\Lambda}}$ and $\underline{\Delta}$ terms equal to, respectively,

$$\underline{\underline{\Lambda}} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \qquad \underline{\Delta} = \begin{bmatrix} 0 \\ 0.2 \\ 0 \end{bmatrix}.$$

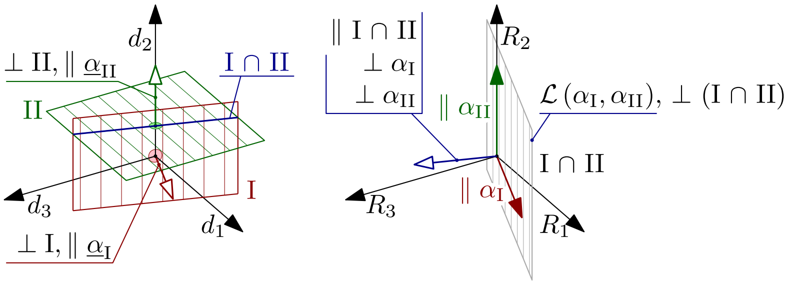
### 0.2.2 General formulation

A set of $m$ constraints

$$d_j = \sum_{d_i \in \underline{d}_{\mathrm{R}}} \lambda_{ji} d_i + \Delta_j \tag{19}$$

[5]i.e. the two subspaces share, with adjusted physical dimensions, the same generator vectors.

[6]alternatively, $d_3$ may be chosen for such a role.




(a) configuration space (b) reaction space

Figure 7: Allowed system configurations and constraint reactions for the pedagogical example of Fig. 6. The allowed displacement sets are easily derived as the homogenous counterpart of (a), and are not represented here.

is defined that states the linear[7] dependence of a partition subset of the $\underline{\mathrm{d}}$ DOFs vector terms, the *tied* ones, on the remaining $d_i$ terms, that retain their independent nature. The independent terms are collected within a reduced cardinality DOF vector $\underline{\mathrm{d}}_{\mathrm{R}}$, and they are referred to as the *retained* ones[8].

Also the inhomogeneous $\Delta_j$ term is provided for in Eqn. 19 to accomodate constraints of the nonzero fixed displacement kind.

The following algebraic relation may then be derived, that defines the initial, unabridged $\underline{\mathrm{d}}$ DOF vector terms based on the subset that produces the retained DOF vector $\underline{\mathrm{d}}_{\mathrm{R}}$

$$\underline{\mathrm{d}} = \underline{\underline{\Lambda}}\, \underline{\mathrm{d}}_{\mathrm{R}} + \underline{\Delta}; \tag{20}$$

the $\underline{\Delta}$ $n$-sized column vector collects the various $\Delta_j$ terms of the 19 constraint equations, and the $n$ rows, $n-m$ columns $\underline{\underline{\Lambda}}$ matrix collects

- the identity relations between corresponding retained DOFs terms that appear in both $\underline{\mathrm{d}}$ and $\underline{\mathrm{d}}_{\mathrm{R}}$, and

[7] more precisely, *linear variation* dependence, due to the presence of the inhomogeneous term.

[8] Here, the definition of the overall, retained, and tied DOF vectors, ($\underline{\mathrm{d}}$, $\underline{\mathrm{d}}_{\mathrm{R}}$, $\underline{\mathrm{d}}_{\mathrm{T}} = \underline{\mathrm{d}} \setminus \underline{\mathrm{d}}_{\mathrm{R}}$, respectively) is overloaded with both its DOF and DOF index (ordered) set counterparts, thus allowing e.g. the $d_i \in \underline{\mathrm{d}}_{\mathrm{R}}$ notation in a vector element context, and the $i \in \underline{\mathrm{d}}_{\mathrm{R}}$ notation in an integer index context.

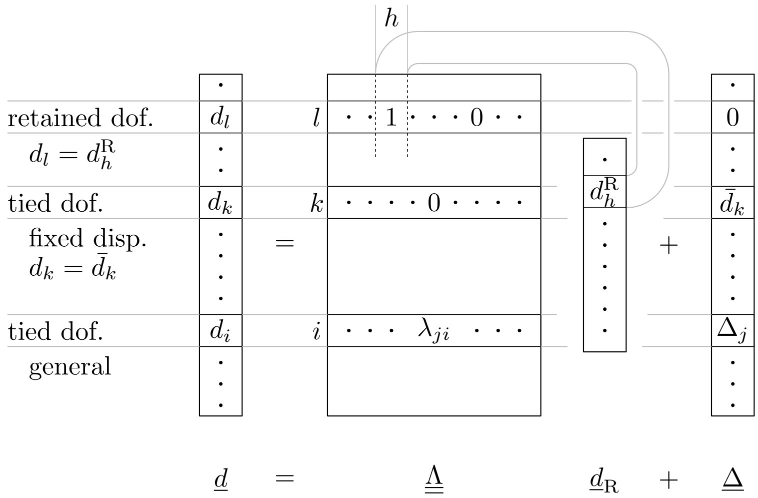



Figure 8: Graphical representation for the $\underline{\underline{\Lambda}}$ matrix in Eq. 20; representative matrix rows are illustrated for a retained DOF, and for two tied DOFs, namely a fixed displacement subcase, and the general case.

- the $\lambda_{ji}$ coefficients that define the linear variation dependence of the tied $d_j$ DOF on the retained $d_i$ DOF.

Figure 8 illustrates a few representatives of the rows whose assembly defines the $\underline{\underline{\Lambda}}$. In the case of a retained global DOF, $d_l$, which finds a counterpart in the $h$-th element of $\underline{\mathrm{d}}_{\mathrm{R}}$, $= d_h^{\mathrm{R}}$, the associated row contains a single unit term at of the intersection of the $l$-th row with the $h$-th column, being zero elsewhere. In the case of a tied DOF of the plain fixed displacement kind (single DOF constraint), the associated row in $\underline{\underline{\Lambda}}$ is null, and the associated inhomogeneous term in $\underline{\Delta}$ equates the imposed value for the displacement. In the case of a tied DOF of the general kind, see Eq. 20, the associated row in the $\underline{\underline{\Lambda}}$ matrix is build upon the $\lambda_{ji}$ linear relation coefficients.

It is finally worth to mention that the virtual displacements in the neighborhood of a feasible constrained configuration are restricted to the linear combinations of the $\underline{\underline{\Lambda}}$ matrix columns $\underline{\Lambda}_j$, i.e.

$$\delta\,\underline{\mathrm{d}} = \underline{\underline{\Lambda}}\,\delta\,\underline{\mathrm{d}}_R = \underline{\Lambda}_1\,\delta\,\underline{\mathrm{d}}_1^R + \underline{\Lambda}_2\,\delta\,\underline{\mathrm{d}}_2^R + \ldots \tag{21}$$

with arbitrary virtual displacement values $\delta\,\underline{\mathrm{d}}_j^R$ for the retained DOF alone.

The ideal constraint hypothesis requires the reaction force vector $\underline{\mathrm{R}}$ to be orthogonal to a generic virtual displacement, and such condition hods if and only if $\underline{\mathrm{R}}$ is orthogonal to each the $\underline{\underline{\Lambda}}$ matrix columns, i.e.

$$\langle\, \underline{\Lambda}_j, \underline{\mathrm{R}} \,\rangle = 0 \quad \forall j, \tag{22}$$

or, equivalently,

$$\underline{\underline{\Lambda}}^\top \underline{\mathrm{R}} = \underline{0}\,. \tag{23}$$

## 0.3 The system of constrained equilibrium equations, and its solution.

The nodal DOF equilibrium equations derived by pairing i) the $\underline{\underline{\mathrm{K}}}\,\underline{\mathrm{d}}$ external forces required to keep the structure in a $\underline{\mathrm{d}}$ deformed configuration, see Eq. 14, ii) the actual external forces $\underline{\mathrm{F}}$ which are applied to the elements as distributed loads, see Eq. 17, or directly at nodes in form of concentrated loads, and iii) the reaction forces $\underline{\mathrm{R}}$ may be cast as

$$\underline{\underline{\mathrm{K}}}\,\underline{\mathrm{d}} = \underline{\mathrm{F}} + \underline{\mathrm{R}}\,. \tag{24}$$

Here, $\underline{\mathrm{d}}$ and $\underline{\mathrm{R}}$ are both unknown.

If constraints are applied, we have

$$\underline{\underline{\mathrm{K}}}\left(\underline{\underline{\Lambda}}\,\underline{\mathrm{d}}_{\mathrm{R}} + \underline{\Delta}\right) = \underline{\mathrm{F}} + \underline{\mathrm{R}} \tag{25}$$

and

$$\underline{\underline{\mathrm{K}}}\,\underline{\underline{\Lambda}}\,\underline{\mathrm{d}}_{\mathrm{R}} = \left(\underline{\mathrm{F}} - \underline{\underline{\mathrm{K}}}\,\underline{\Delta}\right) + \underline{\mathrm{R}}\,, \tag{26}$$

where the inhomogeneous part of the constraint equations is *de facto* assimilated to a further contribution to the external loads.

By projecting the equations on the subspace of allowed configurations

$$\underbrace{\underline{\underline{\Lambda}}^\top \underline{\underline{\mathrm{K}}}\,\underline{\underline{\Lambda}}}_{\underline{\underline{\mathrm{K}}}_{\mathrm{R}}}\,\underline{\mathrm{d}}_{\mathrm{R}} = \underbrace{\underline{\underline{\Lambda}}^\top \left(\underline{\mathrm{F}} - \underline{\underline{\mathrm{K}}}\,\underline{\Delta}\right)}_{\underline{\mathrm{F}}_{\mathrm{R}}} + \underbrace{\underline{\underline{\Lambda}}^\top \underline{\mathrm{R}}}_{=0}, \tag{27}$$

the contribution of the unknown reaction forces, that are normal to such a subspace – see Eq. 23, vanishes.

The linear system of *constrained* nodal DOF equilibrium equations is then set as

$$\underline{\underline{\mathrm{K}}}_{\,\mathrm{R}}\,\underline{\mathrm{d}}_{\,\mathrm{R}} = \underline{\mathrm{F}}_{\,\mathrm{R}} \tag{28}$$

and it may be solved for the retained DOF vector $\underline{\mathrm{d}}_{\,\mathrm{R}}$.

Once the solution vector $\underline{\mathrm{d}}^{\,*}_{\mathrm{R}}$ is found in terms of displacements at retained DOFs, the overall displacement vector and the unknown reaction forces may be derived as

$$\underline{\mathrm{d}}^{\,*} = \underline{\underline{\Lambda}}\,\underline{\mathrm{d}}^{\,*}_{\mathrm{R}} + \underline{\Delta}\,; \tag{29}$$

and

$$\underline{\mathrm{R}}^{\,*} = \underline{\underline{\mathrm{K}}}\,\left(\underline{\underline{\Lambda}}\,\underline{\mathrm{d}}^{\,*}_{\mathrm{R}} + \underline{\Delta}\right) - \underline{\mathrm{F}}\,. \tag{30}$$

Then, for each $j$-th element, the local DOFs vector may be derived based on

$$\underline{\mathrm{d}}^{\,*}_{\mathrm{e}j} = \underline{\underline{\mathrm{P}}}_{\,\mathrm{e}j}\,\underline{\mathrm{d}}^{\,*}, \tag{31}$$

and consequently its in-plane

$$\underline{\epsilon} = \left(\underline{\underline{\mathrm{B}}}^{\,0}_{\mathrm{e}j}(\xi,\eta) + \underline{\underline{\mathrm{B}}}^{\,1}_{\mathrm{e}j}(\xi,\eta)z\right)\,\underline{\mathrm{d}}^{\,*}_{\mathrm{e}j} \tag{32}$$

and out-of-plane strain fields

$$\underline{\bar{\gamma}} = \underline{\underline{\mathrm{B}}}^{\,\bar{\gamma}}_{\mathrm{e}j}(\xi,\eta)\,\underline{\mathrm{d}}^{\,*}_{\mathrm{e}j}, \tag{33}$$

from which the stress components may be easily derived.

### 0.3.1 Rigid body link RBE2

A master (or retained, control, independent, etc.) $C$ node is considered, whose coordinates are defined as $x_C, y_C, z_C$ in a (typically) global reference system, along with a set of $n$ $P_i$ nodes whose coordinates are $x_i, y_i, z_i$.

A kinematic link is to be established such that the DOFs – or a subset of them – associated to the $P_i$ nodes follow the rototranslations of the $C$ control according to the rigid body motion laws.

In the case of a fully constrained $P_i$ node we have

$$\begin{bmatrix} u_i \\ v_i \\ w_i \\ \theta_i \\ \phi_i \\ \psi_i \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & +(z_i - z_C) & -(y_i - y_C) \\ 0 & 1 & 0 & -(z_i - z_C) & 0 & +(x_i - x_C) \\ 0 & 0 & 1 & +(y_i - y_C) & -(x_i - x_C) & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\underline{\underline{\mathbf{L}}}_i} \cdot \begin{bmatrix} u_C \\ v_C \\ w_C \\ \theta_C \\ \phi_C \\ \psi_C \end{bmatrix} \tag{34}$$

where $u, v, w$ $(\theta, \phi, \psi)$ are the translation (rotation) vector components with respect to the $x, y, z$ cartesian reference system. A subset of the above DOF dependency relations may be cast to obtain a partial constraining of the $P_i$ node; a free relative motion of such node with respect to the rigid body is allowed at the unconstrained DOFs.

External actions that are applied to tied $P_i$ DOFs are reduced to the master node in form of a statically equivalent counterpart; the contributions deriving from each $P_i$ node are finally accumulated.