0.1 Advanced modeling tools

0.1.1 Inertia relief

Inertia relief[1] refers to an analysis procedure that allows unconstrained systems – or systems otherwise susceptible to stress-free motions – to be subjected to a quasi-static analysis by taking rigid body inertia forces into account.

Conventional static analysis cannot be performed for such systems since, in the absence of constraints, the stiffness matrix is singular. The structure response is measured relative to a steady state accelerating frame, whose motion is induced by the (usually nonzero) external load resultants.

The inertia relief solution procedure provides for three steps, namely i) the rigid body mode evaluation, ii) the assessment of the inertia relief loads, and iii) the solution of a supported, self-equilibrated static loadcase within the moving frame.

A set of nodal Degree of Freedom (DOF)s is supplied, one each expected rigid body motion, whose *imposed* displacements values uniquely define the structure positioning in space; also, they may be employed in supporting the structure to untangle the stiffness matrix rank-deficiency.

The $\underline{t}_l$ rigid body modes are evaluated by sequentially setting each of these *support* DOF to unity, while retaining the others to zero, and solving for the system of nodal equilibrium equations

$$\underline{\underline{K}}\,\underline{d} = \underline{F}\,, \tag{1}$$

where $\underline{\underline{K}}$ is the structure stiffness matrix, in the absence of further external loads, i.e. $\underline{F} = 0$. Since the tied/retained condition of the structure DOFs does not vary throughout the sequence of aforementioned loadcases, comprised of the final step introduced in the following, a single $\underline{\underline{L}}\,\underline{\underline{L}}^\top$ Cholesky system matrix decomposition is required by the procedure, whose computational burden is thus not significantly increased with respect to the usual static solution.

A rigid body, steady state acceleration field is defined as the linear

[1] XXX some cut and paste from the MSC.Marc vol A manual, please rewrite as required to avoid copyright infringement.

combination of the so defined $\underline{t}_l$ rigid body modes

$$\ddot{\underline{d}} = \underbrace{\begin{bmatrix} \cdots & \underline{t}_l & \cdots \end{bmatrix}}_{\underline{\underline{T}}} \underbrace{\begin{bmatrix} \vdots \\ \alpha_l \\ \vdots \end{bmatrix}}_{\underline{\alpha}}, \tag{2}$$

whose α_l coefficients define the modal acceleration vector $\underline{\alpha}$. Those acceleration terms are then evaluated according to the inertial equilibrium of the structure under the applied $\underline{F}$ external loads, condition, this, that may be stated as

$$\underline{\underline{T}}^\top \underline{\underline{M}}\,\underline{\underline{T}}\,\underline{\alpha} = \underline{\underline{T}}^\top \underline{F} \tag{3}$$

The projection of the equilibrium equations onto the subspace defined by the linear span of the $\underline{t}_l$ rigid body mode vectors – i.e. the left multiplication of both the equation sides by the $\underline{\underline{T}}^\top$ matrix, is solved in place of the overdetermined linear system

$$\underline{\underline{M}}\,\underline{\underline{T}}\,\underline{\alpha} = \underline{F}\ [+\underline{R}_l]$$

since the $\underline{R}_l$ reaction forces associated to the rigid body constraints balance the equilibrium residual components that are orthogonal to such allowed configuration subspace.

The inertia relief forces may then be quantified as $\underline{\underline{M}}\,\underline{\underline{T}}\,\underline{\alpha}$, and superposed to the initial external loads, thus leading to a self equilibrated loading condition in the context of the steady state accelerating frame; by employing the support DOFs to establish a positioning constraint set, the elastic problem may finally be solved in the form

$$\underline{\underline{K}}\,\underline{d} = \underline{F} - \underline{\underline{M}}\,\underline{\underline{T}}\,\underline{\alpha}, \tag{4}$$

The $\underline{d}$ displacement components are expressed with respect to a reference frame that clings to the possibly accelerating structure through the support DOFs; due to the self-equilibrated nature of the applied loads in the moving frame, reaction forces at supports are zero.

As a closing comment, the MSC.Marc solver employs a lumped definition for the system mass matrix for evaluating inertia relief forces.

0.1.2 Harmonic response analysis

The equilibrium equations of a multiple DOF system subject to elastic, inertial and viscous actions may be stated in the general form

$$\underline{\underline{\mathrm{M}}}\,\ddot{\underline{\mathrm{d}}} + \underline{\underline{\mathrm{C}}}\,\dot{\underline{\mathrm{d}}} + \underline{\underline{\mathrm{K}}}\,\underline{\mathrm{d}} = \underline{\mathrm{f}}\,(t), \quad \underline{\mathrm{d}} = \underline{\mathrm{d}}\,(t) \tag{5}$$

where:

- $\underline{\underline{\mathrm{M}}}$ is the mass matrix, which is symmetric and positive definite;
- $\underline{\underline{\mathrm{C}}}$ is the viscous damping matrix, which is symmetric and positive semidefinite;
- $\underline{\underline{\mathrm{K}}}$ is the elastic stiffness matrix, which is symmetric and positive semidefinite: complex terms may appear within the stiffness matrix to represent structural damping contributions;
- $\underline{\mathrm{f}}\,(t)$ is the vector of the external (generalized) forces;
- $\underline{\mathrm{d}}\,(t)$ collects the system DOFs, which vary in time.

The system response is assumed linear – a strong assumption, this, that hardly holds in complex structures as the automotive chassis under scrutiny. The lack of nonlinear analysis tools whose modeling and computational effort is comparable with respect to the one presented in the present section, pushes for some laxity in the linearity prerequisite check, and for the acceptance of a certain extent of error.

The applied force is assumed periodic in time, and so is the long term solution, if linearity holds. Moreover, Fourier decomposition may be applied, and there is no lack in generality in further assuming an harmonic forcing term, and hence an harmonic solution. We have

$$\underline{\mathrm{f}}\,(t) = \frac{\bar{\underline{\mathrm{f}}}\,e^{j\omega t} + \bar{\underline{\mathrm{f}}}^{\,*}e^{-j\omega t}}{2} = \mathrm{Re}(\,\bar{\underline{\mathrm{f}}}\,e^{j\omega t}) \tag{6}$$

where the asterisk superscript denotes the complex conjugate variant of the base vector. We recall that the compact notation

$$\underline{\mathrm{f}}\,(t) = \bar{\underline{\mathrm{f}}}\,e^{j\omega t} \tag{7}$$

extensively employed below defines a complex form for the driving force, whose real part is the portion which is physically applied to the nodes over time, i.e.

$$\mathrm{Re}(\underline{\bar{\mathrm{f}}}\, e^{j\omega t}) = \mathrm{Re}(\underline{\bar{\mathrm{f}}}\,) \cos\omega t - \mathrm{Im}(\underline{\bar{\mathrm{f}}}\,) \sin\omega t \tag{8}$$

This compact formalism is not rigorous but still it is effective, and hence commonly employed. Any phase difference amongst the applied nodal excitations may be described by resorting to the complex nature of the $\underline{\bar{\mathrm{f}}}$ vector terms.

In the neglection of the transient response, the harmonic tentative solution

$$\underline{\mathrm{d}}\,(t) = \underline{\bar{\mathrm{d}}}\, e^{j\omega t} \tag{9}$$

is substituted within Eq. 5, thus obtaining

$$\left(-\omega^2 \underline{\underline{\mathrm{M}}} + j\omega\, \underline{\underline{\mathrm{C}}} + \underline{\underline{\mathrm{K}}}\right) \underline{\bar{\mathrm{d}}} = \underline{\bar{\mathrm{f}}} \tag{10}$$

where the $e^{j\omega t}$ time varying, generally nonzero factors are simplified away.

Expression 11 defines a system of linear complex equations, one each DOF, in the complex unknown vector $\underline{\bar{\mathrm{d}}}$; equivalently, each complex equation and each unknown term may be split into the associated real and imaginary parts, thus leading to a system of linear, real equations whose order is twice the number of the discretized structure DOFs.

The system matrix varies with the ω parameter, and in particular its stiffness contribute $\underline{\underline{\mathrm{K}}}$ is dominant for low ω values, whereas the $\underline{\underline{\mathrm{C}}}, \underline{\underline{\mathrm{M}}}$ terms acquire relevance with growing ω.

In distributed inertia systems, however, it is a misleading claim that the stiffness matrix contribution becomes negligible with high ω values, since – with the notable exception of external loads that are directly applied to concentrated masses or rigid bodies – the pulsation is unphysically high above which such behaviour arises.

Since Eqns. 11 are independently solved for each ω value, it constitutes no added complexity to let $\underline{\underline{\mathrm{M}}}$, $\underline{\underline{\mathrm{C}}}$, $\underline{\underline{\mathrm{K}}}$ and $\underline{\bar{\mathrm{f}}}$ vary according to the same parameter.

Finally, in the absence of the damping-related imaginary terms within the system matrix, the Eq. 11 problem algebraic order is led back to the bare number of system DOFs; in fact, two independent real system of equations – that share a common $\underline{\underline{L}}\,\underline{\underline{L}}^\top$ matrix decomposition – may be cast for the real and the imaginary parts of $\underline{\bar{d}}$ and $\underline{\bar{f}}$.

0.1.3 Modal analysis

The present paragraph briefly deals with the structure's natural modes, i.e. those periodic[2] motions that are allowed according to Eq. 5, in the further absence of externally applied loads.

A necessary condition for a motion to endure in the absence of a driving load is the absence of dissipative phenomena; it is hence necessary to have a zero $\underline{\underline{C}}$ damping matrix, whereas the $\underline{\underline{K}}$ stiffness matrix must be free of imaginary terms. This hypothesis holding, Eq. 11 is reduced to the following real-term algebraic form

$$\left(-\omega^2\,\underline{\underline{M}} + \underline{\underline{K}}\right)\,\underline{\bar{d}} = \underline{0} \tag{11}$$

whose nontrivial solutions constitute a set of $(\omega_i^2,\,\underline{\hat{d}}_i)$ *generalized* eigenvalue/eigenvector pairs, one each system DOF, if eigenvalue multiplicity is taken into account.

In the context of each $(\omega_i^2,\,\underline{\hat{d}}_i)$ pair, ω_i is the natural pulsation ($\omega_i = 2\pi f_i$, where f_i is the natural frequency), whereas the $\underline{\hat{d}}_i$ vector of generalized displacemts is named natural mode.

The extraction of the Eq. 11 nontrivial solutions reduces to a *standard* eigenvalue problem is the algebraic form is left-multiplied by the mass matrix inverse, i.e.

$$\left(\underline{\underline{M}}^{-1}\,\underline{\underline{K}} - \omega^2\,\underline{\underline{I}}\right)\,\underline{\hat{d}} = \underline{0}\,; \tag{12}$$

the availability of solvers that specifically approach the generalized problem avoid such computationally uneconomical preliminary.

It is worth to recall that in the case of eigenvalues with non-unit multiplicity – concept, this, that is to be contextualized within the limited precision floating point arithmetics[3] – the associated eigenvectors must be considered only through their linear combination; the specific selection of the base elements for representing such a subspace (i.e.,

[2] *harmonic* in the context of linearly behaving systems
[3] XXX

each single eigenvector) derives in fact from the unpredictable interaction between the truncation error and the inner mechanics of the numerical procedure.

Also, the eigenvectors that are associated to eigevalues of unit multiplicity are returned by the numerical solver in the misleading form of a definite vector, whereas an arbitrary (both in sign and magnitude) scaling factor has to be prepended.

In particular, any speculation which is not robust with respect to such arbitrary scaling (or combination) is of no engineering relevance, and must be avoided.

Finally, in continuous elasticity, no upper bound exists for natural frequencies; in Finite Element (FE) discretized structure, an apparent upper bound exists, which depends on local element size[4].

A common normalizing rule for the natural modes is the one that produces a unit modal mass m_i, i.e.

$$m_i = \underline{\hat{\mathrm{d}}}_i^\top \, \underline{\underline{\mathrm{M}}} \, \underline{\hat{\mathrm{d}}}_i = 1 \tag{13}$$

this rule is e.g. adopted by the MSC.Marc solver in its default configuration.

The resonant behaviour of the system in correspondence with a natural frequency may be investigated by substituting the following tentative solution

$$\underline{\mathrm{x}}(t) = a \, \underline{\hat{\mathrm{d}}}_i \sin(\omega_i t) \tag{14}$$

within the dynamic equilibrium equations 5, with

$$f(t) = \underline{\hat{\mathrm{f}}} \cos(\omega_i t), \tag{15}$$

and thus obtaining

$$\underbrace{\left(-\omega_i^2 \, \underline{\underline{\mathrm{M}}} + \underline{\underline{\mathrm{K}}}\right) \underline{\hat{\mathrm{d}}}_i}_{=\underline{0}} a_i \sin(\omega_i t) + \omega_i a_i \, \underline{\underline{\mathrm{C}}} \, \underline{\hat{\mathrm{d}}}_i \cos(\omega_i t) = \underline{\bar{\mathrm{f}}} \cos(\omega_i t). \tag{16}$$

By simplifying away the generally nonzero time modulating factors, and by left-multiplying both equation sides by $\underline{\hat{\mathrm{d}}}^\top$ – i.e. by projecting

[4] In particular, the natural oscillation period for the highest dynamic mode is estimated with order of magnitude precision as the minimum time it takes a pressure wave to travel between two different nodes in the discretized structure.

the equation residual along the subspace defined by the eigenvector itself, we obtain an amplitude term in the form

$$a_i = \frac{\underline{\hat{\mathrm{d}}}^\top \underline{\bar{\mathrm{f}}}}{\omega_i \underline{\hat{\mathrm{d}}}^\top \underline{\underline{\mathrm{C}}}\, \underline{\hat{\mathrm{d}}}_i} \tag{17}$$

whose singularity is prevented only a) in the presence of a damping matrix that associates nonzero and non-orthogonal viscous reactions to the motion described by the natural mode under scrutiny, or b) if the driving load is orthogonal to such natural mode, i.e. it unable to perform periodic work on such a motion. The nature of the expression 17 numerator will be further discussed in the following paragraph.

0.1.4 Harmonic response through mode superposition

In the case the eigenvalues associated with the dynamic modes are all distinct[5], the following orthogonality conditions hold

$$\underline{\hat{\mathrm{d}}}_j^\top \underline{\underline{\mathrm{M}}}\, \underline{\hat{\mathrm{d}}}_i = m_i \delta_{ij} \qquad\qquad \underline{\hat{\mathrm{d}}}_j^\top \underline{\underline{\mathrm{K}}}\, \underline{\hat{\mathrm{d}}}_i = m_i \omega_i^2 \delta_{ij} \tag{18}$$

where δ_{ij} is the Kroneker delta function, and $m_i = 1$ is the i-th modal mass, which is unitary due to the the $\underline{\hat{\mathrm{d}}}_i$ normalization.

It is further assumed that it is possible to describe the elastic body motion through a linear combination of a (typically narrow) subset of the dynamic natural modes. Such assumption may be rationalized in two equivalent ways: on one hand, the contribution of the neglected modes is assumed negligible, and hence ignored; on the other hand, it is imagined that a set of kinematic constraints is imposed, that rigidly impede any additional system motion with respect to the chosen set. According to this latter explanation, reaction forces will be raised that absorb any equilibrium residual term which is orthogonal with respect to the allowed displacements.

The subset defined by the first m eigenvectors $(1 \leq m \ll n)$ are commonly employed, whereas different assortments are possible; a control calculation perfomed with a wider base may be employed for error estimation.

[5]condition, this, that is assumed to hold; a slightly perturbed FE discretization may be effective in separating the instances of a theoretically multiple natural frequency.

By stacking those first m normalized column eigenvectors into the $\underline{\underline{\Xi}}$ matrix below,

$$\underline{\underline{\Xi}} = \left[\underline{\hat{\mathrm{d}}}_1 \quad \cdots \quad \underline{\hat{\mathrm{d}}}_l \quad \cdots \quad \underline{\hat{\mathrm{d}}}_m \right], \tag{19}$$

any $\underline{\bar{\mathrm{d}}}$ configuration belonging to the linear span of the selected modes may be expressed through a vector of m *modal* coordinates $\bar{\xi}_l$, as in

$$\underline{\bar{\mathrm{d}}} = \underline{\underline{\Xi}}\, \underline{\bar{\xi}} \tag{20}$$

.

Due to the natural modes orthogonality conditions 18, the $\underline{\underline{\Xi}}$ tranformation matrix diagonalizes both the mass and the stiffness matrices, since

$$\underline{\underline{\Xi}}^\top \underline{\underline{\mathrm{M}}}\, \underline{\underline{\Xi}} = \underline{\underline{\mathrm{I}}} \qquad\qquad \underline{\underline{\Xi}}^\top \underline{\underline{\mathrm{K}}}\, \underline{\underline{\Xi}} = \underline{\underline{\Lambda}} = \mathrm{diag}(\omega_l^2); \tag{21}$$

by appling such transformation to the damping matrix, however, a dense matrix is generally obtained.

The *Rayleigh* or *proportional* damping matrix definition assumes that the latter may be passably represented as a linear combination of the mass matrix and of the stiffness matrix: in particular

$$\underline{\underline{\mathrm{C}}} = \alpha\, \underline{\underline{\mathrm{M}}} + \beta\, \underline{\underline{\mathrm{K}}} \tag{22}$$

where α and β are commonly named *mass* and *stiffness matrix multipliers*, respectively; according to such assumption, the damping matrix is also diagonalized by the $\underline{\underline{\Xi}}$ tranformation matrix.

Equation 11 algebraic problem may be cast in terms of the m ξ_l modal unknowns, thus obtaining

$$\underline{\underline{\Xi}}^\top \left(-\omega^2\, \underline{\underline{\mathrm{M}}} + j\omega\, \underline{\underline{\mathrm{C}}} + \underline{\underline{\mathrm{K}}} \right) \underline{\underline{\Xi}}\, \underline{\bar{\xi}} = \underline{\underline{\Xi}}^\top \underline{\bar{\mathrm{f}}} \tag{23}$$

which reduces to the diagonal form

$$\left(-\omega^2\, \underline{\underline{\mathrm{I}}} + j\omega \left(\alpha\, \underline{\underline{\mathrm{I}}} + \beta\, \underline{\underline{\Lambda}} \right) + \underline{\underline{\Lambda}} \right) \underline{\bar{\xi}} = \underline{\underline{\Xi}}^\top \underline{\bar{\mathrm{f}}}, \tag{24}$$

or, equivalently, to the set of m independent complex equations

$$\left(-\omega^2 + j\omega \left(\alpha + \beta\omega_l^2 \right) + \omega_l^2 \right) \xi_l = q_l, \quad j = 1 \ldots m \tag{25}$$

where $q_l = \left\langle \underline{\hat{\mathrm{d}}}_l, \underline{\bar{\mathrm{f}}} \right\rangle$ is the coupling factor between the external load and the j-th natural mode.

The algebraic equation above may be interpreted as the characteristic equation of an harmonically driven single DOF oscillator that exhibits the following properties:

- its mass is unity;
- its natural frequency equals that of the j-th natural mode;
- its damping ratio ζ_l is a combination of the two Rayleigh damping coefficients, i.e.
$$\zeta_l = \frac{1}{2}\left(\frac{\alpha}{\omega_l} + \beta\omega_l\right);$$
- the external load real(imaginary) term is defined as the cyclic work that the external load performs upon a system motion described as the sinusoidal (cosinusoidal) modulation in time of the j-th modal shape, divided by π.

The uncoupled equations 25 may be solved resorting to complex division arithmetics, thus leading to the definition of the $\bar{\xi}_l$ modal amplitude and phase terms; in particular we have that the j-th modal shape is modulated in time according to the function

$$\begin{aligned}\xi_l(t) &= \mathrm{Re}(\bar{\xi}_l)\cos\omega t - \mathrm{Im}(\bar{\xi}_l)\sin\omega t \\ &= \left|\bar{\xi}_l\right|\cos\left(\omega t + \psi_l - \phi_l\right)\end{aligned}$$

whose terms are detailed in the following.

The auxiliary parameters

$$a_l = 1 - r_l^2 \qquad b_l = 2\zeta_l r_l \qquad r_l = \frac{\omega}{\omega_l}$$

are first defined; we then have the oscillation amplitude and phase terms

$$\begin{aligned}\left|\bar{\xi}_l\right| &= \frac{|\bar{q}_l|}{\omega_l^2}\frac{1}{\sqrt{a_l^2 + b_l^2}} \\ \psi_l &= \arg(\bar{q}_l) \\ \phi_l &= \arg(a_l + jb_l)\end{aligned}$$

or, equivalently, the real and imaginary parts

$$\begin{aligned}\mathrm{Re}(\bar{\xi}_l) &= \frac{1}{\omega_l^2}\frac{a_l\,\mathrm{Re}(\bar{q}_l) + b_l\,\mathrm{Im}(\bar{q}_l)}{a_l^2 + b_l^2}\\ \mathrm{Im}(\bar{\xi}_l) &= \frac{1}{\omega_l^2}\frac{a_l\,\mathrm{Im}(\bar{q}_l) - b_l\,\mathrm{Re}(\bar{q}_l)}{a_l^2 + b_l^2}.\end{aligned}$$