$$N = \int_{\mathcal{A}} \sigma_{zz} d\mathcal{A}$$

$$Q_y = \int_{\mathcal{A}} \tau_{yz} d\mathcal{A}$$

$$Q_x = \int_{\mathcal{A}} \tau_{zx} d\mathcal{A}$$

$$M_x \equiv M_{(G,x)} = \int_{\mathcal{A}} \sigma_{zz} y d\mathcal{A}$$

$$M_y \equiv M_{(G,y)} = -\int_{\mathcal{A}} \sigma_{zz} x d\mathcal{A}$$

$$M_t \equiv M_{(C,z)} = \int_{\mathcal{A}} \left[\tau_{yz}(x - x_C) - \tau_{zx}(y - y_C) \right] d\mathcal{A}$$

(a)

(b)

(c)

(d)

(e)
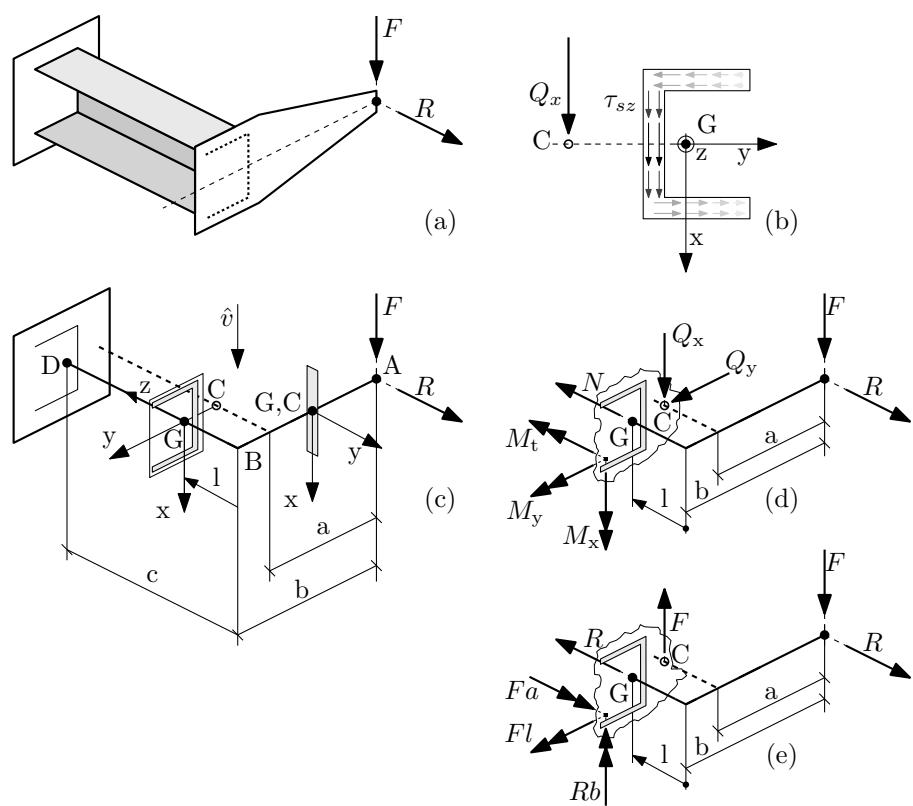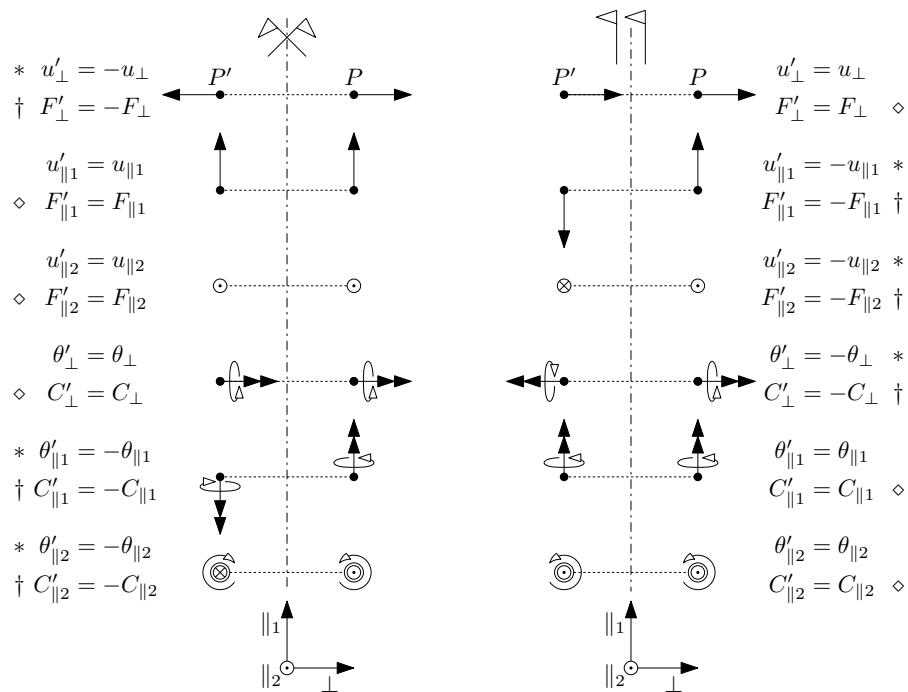
Figure 1: An overview of symmetrical and skew-symmetrical (generalized) loading and displacements.

(a)   (b)   (c)

(d)   (e)   (f)
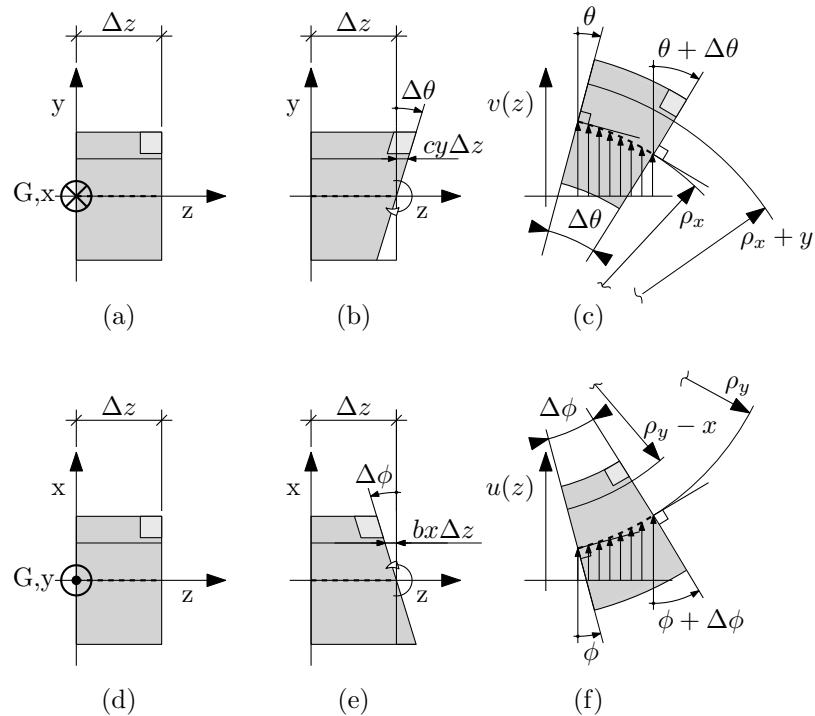
$$\epsilon_z = a + bx + cy \tag{1}$$

$$\frac{d\theta}{dz} = \frac{1}{\rho_x}, \quad \theta = -\frac{dv}{dz}, \quad \frac{d^2v}{dz^2} = -\frac{1}{\rho_x} \tag{2}$$

$$\frac{d\phi}{dz} = \frac{1}{\rho_y}, \quad \phi = +\frac{du}{dz}, \quad \frac{d^2u}{dz^2} = +\frac{1}{\rho_y} \tag{3}$$

$$\sigma_z = E_z\epsilon_z = E_z\left(\bar{\epsilon} - \frac{1}{\rho_y}x + \frac{1}{\rho_x}y\right) \tag{4}$$

$$N = \iint_{\mathcal{A}} E_z\epsilon_z dA = \overline{EA}\bar{\epsilon} \tag{5}$$

$$M_x = \iint_{\mathcal{A}} E_z\epsilon_z y dA = \overline{EJ}_{xx}\frac{1}{\rho_x} - \overline{EJ}_{xy}\frac{1}{\rho_y} \tag{6}$$

$$M_y = -\iint_{\mathcal{A}} E_z\epsilon_z x dA = -\overline{EJ}_{xy}\frac{1}{\rho_x} + \overline{EJ}_{yy}\frac{1}{\rho_y} \tag{7}$$
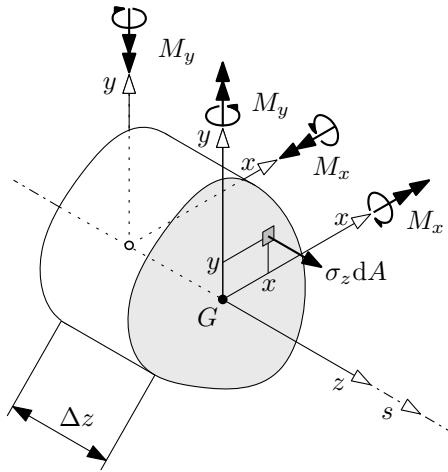
$$\overline{EA} = \iint_{\mathcal{A}} E_z(x,y) \, dA \tag{8}$$

$$\overline{EJ}_{xx} = \iint_{\mathcal{A}} E_z(x,y) yy \, dA \tag{9}$$

$$\overline{EJ}_{xy} = \iint_{\mathcal{A}} E_z(x,y) yx \, dA \tag{10}$$

$$\overline{EJ}_{yy} = \iint_{\mathcal{A}} E_z(x,y) xx \, dA \tag{11}$$

$$\bar{\epsilon} = \frac{N}{\overline{EA}}. \tag{12}$$

$$\frac{1}{\rho_x} = \frac{M_x \overline{EJ}_{yy} + M_y \overline{EJ}_{xy}}{\overline{EJ}_{xx}\overline{EJ}_{yy} - \overline{EJ}_{xy}^2} \tag{13}$$

$$\frac{1}{\rho_y} = \frac{M_x \overline{EJ}_{xy} + M_y \overline{EJ}_{xx}}{\overline{EJ}_{xx}\overline{EJ}_{yy} - \overline{EJ}_{xy}^2} \tag{14}$$

$$\frac{1}{\rho_{\text{eq}}} = \sqrt{\frac{1}{\rho_x^2} + \frac{1}{\rho_y^2}} \tag{15}$$

$$\sigma_z = E_z \epsilon_z \tag{16}$$

$$= \alpha M_x + \beta M_y + \gamma N \tag{17}$$
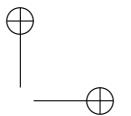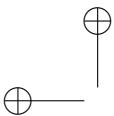
$$\alpha\left(x, y, E_z, \overline{EJ}_{**}\right) = E_z(x, y) \frac{-\overline{EJ}_{xy}x + \overline{EJ}_{yy}y}{\overline{EJ}_{xx}\overline{EJ}_{yy} - \overline{EJ}_{xy}^2} \tag{18}$$

$$\beta\left(x, y, E_z, \overline{EJ}_{**}\right) = E_z(x, y) \frac{-\overline{EJ}_{xx}x + \overline{EJ}_{xy}y}{\overline{EJ}_{xx}\overline{EJ}_{yy} - \overline{EJ}_{xy}^2} \tag{19}$$

$$\gamma\left(x, y, E_z, \overline{EA}\right) = E_z(x, y) \frac{1}{\overline{EA}}. \tag{20}$$

$$(x_N, y_N) \equiv \left(\frac{\bar{e}\rho_x^2\rho_y}{\rho_x^2 + \rho_y^2}, -\frac{\bar{e}\rho_x\rho_y^2}{\rho_x^2 + \rho_y^2} \right);$$

$$\hat{n}_{\|} = \sqrt{\rho_x^2 + \rho_y^2}\left(\frac{1}{\rho_x}, \frac{1}{\rho_y} \right),$$

$$\hat{n}_{\perp} = \sqrt{\rho_x^2 + \rho_y^2}\left(-\frac{1}{\rho_y}, \frac{1}{\rho_x} \right),$$

$$\lambda \begin{bmatrix} M_x \\ M_y \end{bmatrix} = \begin{bmatrix} \frac{1}{\rho_x} \\ \frac{1}{\rho_y} \end{bmatrix} = \underbrace{\frac{1}{\overline{EJ}_{xx}\overline{EJ}_{yy} - \overline{EJ}_{xy}^2} \begin{bmatrix} \overline{EJ}_{yy} & \overline{EJ}_{xy} \\ \overline{EJ}_{xy} & \overline{EJ}_{xx} \end{bmatrix}}_{[\overline{EJ}]} \begin{bmatrix} M_x \\ M_y \end{bmatrix} \tag{21}$$
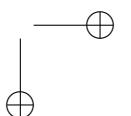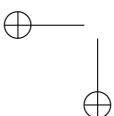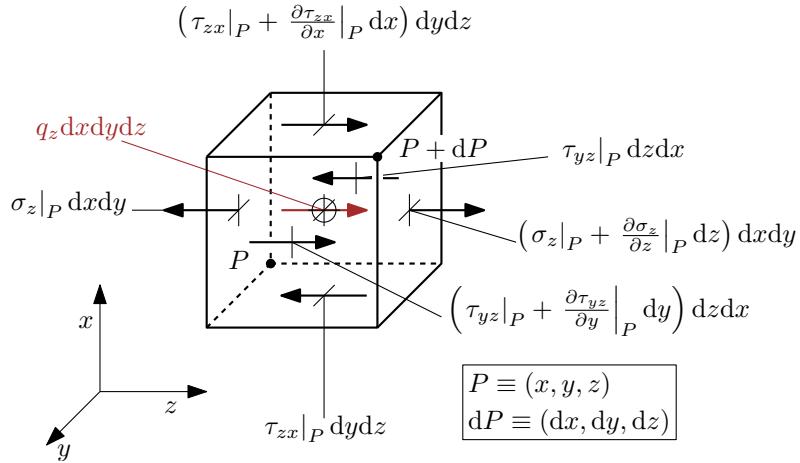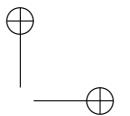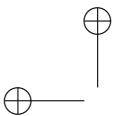
$$S_y = \frac{dM_x}{dz}, \quad S_x = -\frac{dM_y}{dz}, \tag{22}$$

$$\frac{d\sigma_z}{dz} = \alpha\left(x, y, E_z, \overline{EJ}_{**}\right) S_y - \beta\left(x, y, E_z, \overline{EJ}_{**}\right) S_x \tag{23}$$

$$\frac{d\tau_{zx}}{dx} + \frac{d\tau_{yz}}{dy} + \frac{d\sigma_z}{dz} + q_z = 0 \tag{24}$$

$$\bar{\tau}_{zi} t = \int_{A^*} \frac{d\sigma_z}{dz} dA, \tag{25}$$

$$\bar{\tau}_{zi} = \frac{1}{t} \int_t \tau_{zi} dr \tag{26}$$

$$\bar{\tau}_{zi} t = \int_{A^*} \left(\frac{yS_y}{J_{xx}} + \frac{xS_x}{J_{yy}}\right) dA = \frac{\bar{y}^* A^*}{J_{xx}} S_y + \frac{\bar{x}^* A^*}{J_{yy}} S_x, \tag{27}$$

$$\bar{\tau}_{zi} t = q_{zi} = \int_0^s \int_{-t/2}^{t/2} \frac{d\sigma_z}{dz} dr d\varsigma \approx \int_0^s \left.\frac{d\sigma_z}{dz}\right|_{r=0} t d\varsigma. \tag{28}$$

$$\bar{\tau}_{zi}(s) t(s) = q(s) = \int_a^s \frac{d\sigma_z}{dz} t d\varsigma + \underbrace{\bar{\tau}_{zi}(a) t(a)}_{q_A}. \tag{29}$$

$$\tau(s) = \frac{S_1}{\mathcal{A}} f_{;S1}(s) + \frac{S_2}{\mathcal{A}} f_{;S2}(s) + \tau_A f_{;A}(s) + \tau_B f_{;B}(s) \tag{30}$$

dz

t

A^*

$\bar{\tau}_{zi}$

D

$d\sigma_z$

$d\sigma_z = 0$

G

z

M_y

M_x

S_x

S_y

S_y

S_x

$M_x + dM_x$

$M_y + dM_y$

t

A^*

$\bar{\tau}_{zi}$

t

A^*

$\bar{\tau}_{zi}$

$f_{;S1}(s)$

$f_{;S2}(s)$

$f_{;A}(s)$

$f_{;B}(s)$

A

B

S_1^{u}

S_2^{u}

τ_A^{u}

τ_B^{u}

$\equiv [\,1 \text{ stress unit }]$

(a)

(b)

(c)

(d)
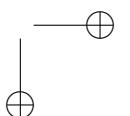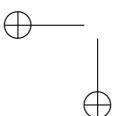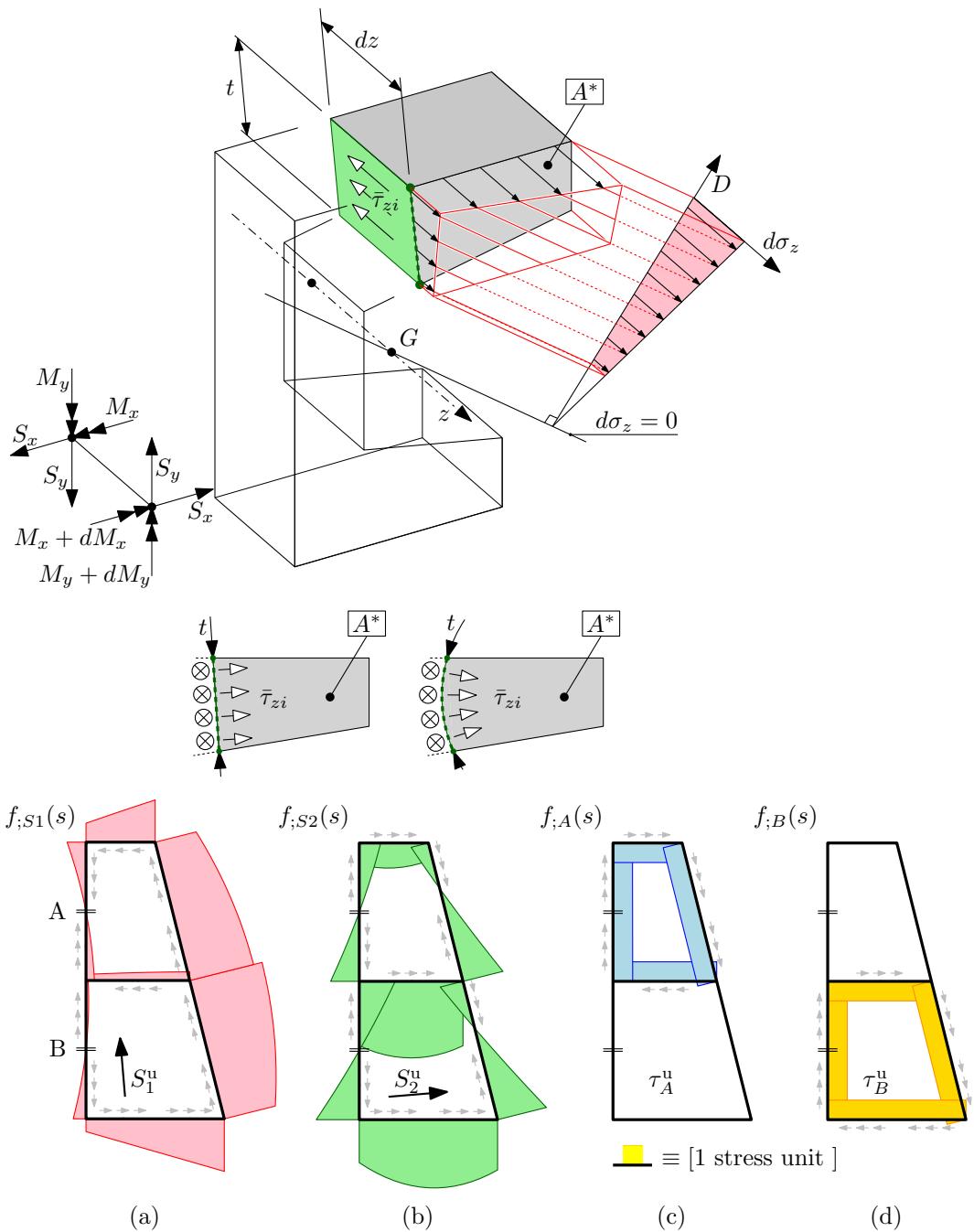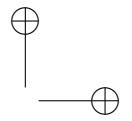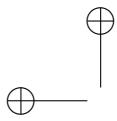
$$\Delta U = \int_s \frac{\tau^2}{2G_{sz}} t \Delta z ds \tag{31}$$

$$\frac{\partial \Delta U}{\partial \bar{\tau}_i} = \bar{\delta}_i t \Delta z \tag{32}$$

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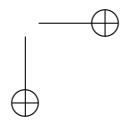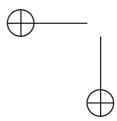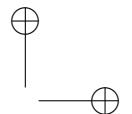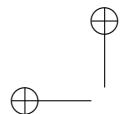
$$K_t = \frac{4A^2}{\oint \frac{1}{t}dl} \tag{33}$$

$$\tau_{\max} = \frac{M_t}{2t_{\min}A} \tag{34}$$

$$K_T \approx \frac{1}{3}\int_0^l t^3(s)ds \tag{35}$$

$$K_T \approx \frac{1}{3}\sum_i l_i t_i^3 \tag{36}$$

$$\tau_{\max} = \frac{M_t t_{\max}}{K_T} \tag{37}$$
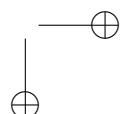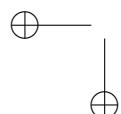
$$q_i = \frac{\partial U}{\partial Q_i}$$

$$\frac{dU}{dl} = \frac{1}{2} \begin{pmatrix} N \\ M_{\mathrm{x}} \\ M_{\mathrm{y}} \\ Q_{\mathrm{x}} \\ Q_{\mathrm{y}} \\ M_{\mathrm{t}} \end{pmatrix}^{\top} \begin{pmatrix} a_{1,1} & g_{1,2} & g_{1,3} & i_{1,4} & i_{1,5} & i_{1,6} \\ 0 & b_{2,2} & e_{2,3} & i_{2,4} & i_{2,5} & i_{2,6} \\ 0 & 0 & b_{3,3} & i_{3,4} & i_{3,5} & i_{3,6} \\ 0 & 0 & 0 & c_{4,4} & f_{4,5} & h_{4,6} \\ 0 & 0 & 0 & 0 & c_{5,5} & h_{5,6} \\ 0 & 0 & 0 & 0 & 0 & d_{6,6} \end{pmatrix} \begin{pmatrix} N \\ M_{\mathrm{x}} \\ M_{\mathrm{y}} \\ Q_{\mathrm{x}} \\ Q_{\mathrm{y}} \\ M_{\mathrm{t}} \end{pmatrix}, \quad (38)$$

$$a_{1,1} = \frac{1}{EA} \qquad \{b_{2,2}, b_{3,3}, e_{2,3}\} = \frac{\{J_{\mathrm{yy}}, J_{\mathrm{xx}}, 2J_{\mathrm{xy}}\}}{E\left(J_{\mathrm{xx}}J_{\mathrm{yy}} - J_{\mathrm{xy}}^2\right)}$$

$$d_{6,6} = \frac{1}{GK_t} \qquad \{c_{4,4}, c_{5,5}, f_{4,5}\} = \frac{\{\chi_{\mathrm{x}}, \chi_{\mathrm{y}}, \chi_{\mathrm{xy}}\},}{GA}$$
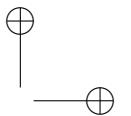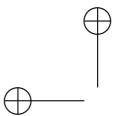
$$u_P = u + z\left(1 + \check{\epsilon}_z\right)\frac{\cos\theta}{\sqrt{1 - \sin^2\phi\sin^2\theta}}\sin\phi$$

$$v_P = v - z\left(1 + \check{\epsilon}_z\right)\frac{\cos\phi}{\sqrt{1 - \sin^2\phi\sin^2\theta}}\sin\theta$$

$$w_P = w + z\left(\left(1 + \check{\epsilon}_z\right)\frac{\cos\phi\cos\theta}{\sqrt{1 - \sin^2\phi\sin^2\theta}} - 1\right),$$

$$\check{\epsilon}_z(z) = \frac{1}{z}\int_0^z \epsilon_z d\varsigma$$

$$= \frac{1}{z}\int_0^z -\frac{\nu}{1-\nu}\left(\epsilon_x + \epsilon_y\right)d\varsigma,$$

$$u_P = u + z\phi \tag{39}$$

$$v_P = v - z\theta \tag{40}$$

$$w_P = w. \tag{41}$$

$$\frac{\partial w}{\partial x} = \bar{\gamma}_{zx} - \phi \tag{42}$$

$$\frac{\partial w}{\partial y} = \bar{\gamma}_{yz} + \theta \tag{43}$$

$$\epsilon_x = \frac{\partial u_P}{\partial x} = \frac{\partial u}{\partial x} + z\frac{\partial\phi}{\partial x} \tag{44}$$

$$\epsilon_y = \frac{\partial v_P}{\partial y} = \frac{\partial v}{\partial y} - z\frac{\partial\theta}{\partial y} \tag{45}$$

$$\gamma_{xy} = \frac{\partial u_P}{\partial y} + \frac{\partial v_P}{\partial x} \tag{46}$$

$$= \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + z\left(+\frac{\partial\phi}{\partial y} - \frac{\partial\theta}{\partial x}\right) \tag{47}$$

$$\underline{e} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} = \begin{bmatrix} e_x \\ e_y \\ g_{xy} \end{bmatrix} \equiv \underline{\epsilon}\,\mathrm{Q} \tag{48}$$

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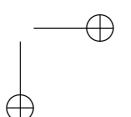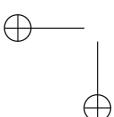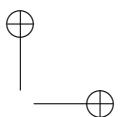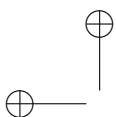
$$\underline{\kappa} = \begin{bmatrix} +\frac{\partial \phi}{\partial x} \\ -\frac{\partial \theta}{\partial y} \\ +\frac{\partial \phi}{\partial y} - \frac{\partial \theta}{\partial x} \end{bmatrix} = \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \tag{49}$$

$$\underline{\epsilon}_P \equiv \underline{\epsilon} = \underline{e} + z\underline{\kappa}. \tag{50}$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \underline{\sigma} = \underline{\underline{D}}\,\underline{\epsilon} = \underline{\underline{D}}\,\underline{e} + z\underline{\underline{D}}\,\underline{\kappa}, \tag{51}$$

$$\underline{\underline{D}} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}, \tag{52}$$

$$\epsilon_z = -\frac{\nu}{1-\nu}\left(\epsilon_x + \epsilon_y\right). \tag{53}$$

$$\underline{q} = \begin{bmatrix} q_x \\ q_y \\ q_{xy} \end{bmatrix} = \int_h \underline{\sigma}\,dz$$
$$= \underbrace{\int_h \underline{\underline{D}}\,dz}_{\underline{\underline{a}}}\,\underline{e} + \underbrace{\int_h \underline{\underline{D}}\,z\,dz}_{\underline{\underline{b}}}\,\underline{\kappa} \tag{54}$$

$$\underline{q}_z = \begin{bmatrix} q_{xz} \\ q_{yz} \end{bmatrix} \qquad q_{xz} = \int_h \tau_{zx}\,dz \qquad q_{yz} = \int_h \tau_{yz}\,dz. \tag{55}$$

$$\underline{m} = \begin{bmatrix} m_x \\ m_y \\ m_{xy} \end{bmatrix} = \int_h \underline{\sigma}\,z\,dz$$
$$= \underbrace{\int_h \underline{\underline{D}}\,z\,dz}_{\underline{\underline{b}} \equiv \underline{\underline{b}}^{\mathrm{T}}}\,\underline{e} + \underbrace{\int_h \underline{\underline{D}}\,z^2\,dz}_{\underline{\underline{c}}}\,\underline{\kappa}. \tag{56}$$

$$\begin{bmatrix} \underline{q} \\ \underline{m} \end{bmatrix} = \begin{bmatrix} \underline{\underline{a}} & \underline{\underline{b}} \\ \underline{\underline{b}}^{\mathrm{T}} & \underline{\underline{c}} \end{bmatrix} \begin{bmatrix} \underline{e} \\ \underline{\kappa} \end{bmatrix} \tag{57}$$

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$$v^\dagger = \frac{1}{2} \left[\begin{array}{c} \underline{q} \\ \hline \underline{m} \end{array} \right]^\top \left[\begin{array}{c} \underline{e} \\ \hline \underline{\kappa} \end{array} \right] \tag{58}$$

$$= \frac{1}{2} \left[\begin{array}{c} \underline{e} \\ \hline \underline{\kappa} \end{array} \right]^\top \left[\begin{array}{cc} \underline{\underline{a}} & \underline{\underline{b}} \\ \underline{\underline{b}}^\top & \underline{\underline{c}} \end{array} \right] \left[\begin{array}{c} \underline{e} \\ \hline \underline{\kappa} \end{array} \right] . \tag{59}$$

$$\underline{\underline{a}} = h\,\underline{\underline{D}} \qquad\qquad \underline{\underline{b}} = \underline{\underline{0}} \qquad\qquad \underline{\underline{c}} = \frac{h^3}{12}\underline{\underline{D}},$$

$$\underline{\gamma}_z = \left[\begin{array}{c} \bar{\gamma}_{yz} \\ \bar{\gamma}_{zx} \end{array} \right]$$

$$\underline{q}_z = \left[\begin{array}{c} q_{xz} \\ q_{yz} \end{array} \right]$$

$$v^\ddagger = \frac{1}{2}\,\underline{q}_z^\top\,\underline{\gamma}_z = \frac{1}{2}q_{xz}\bar{\gamma}_{xz} + \frac{1}{2}q_{yz}\bar{\gamma}_{yz}. \tag{60}$$

$$v^\ddagger = \frac{1}{2}\,\underline{\gamma}_z^\top\,\chi \underbrace{\int_h \underline{\underline{G}}\,dz}_{\underline{\underline{\Gamma}}}\,\underline{\gamma}_z \tag{61}$$

$$\underline{q}_z = \underline{\underline{\Gamma}}\,\underline{\gamma}_z. \tag{62}$$

$$\underline{\underline{G}} = \frac{E}{2\,(1+\nu)} \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right],$$

$$\tau_{zx}(z) = -\int_{-\frac{h}{2}+o}^{z} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}dz \tag{63}$$

$$\tau_{yz}(z) = -\int_{-\frac{h}{2}+o}^{z} \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y}dz. \tag{64}$$

$$\underline{\underline{D}}_{123} = \left[\begin{array}{ccc} \frac{E_1}{1-\nu_{12}\nu_{21}} & \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} & 0 \\ \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & \frac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{array} \right] \tag{65}$$

$$\left[\begin{array}{c} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{array} \right] = \underline{\underline{T}}_1 \left[\begin{array}{c} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{array} \right] \qquad\qquad \left[\begin{array}{c} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{array} \right] = \underline{\underline{T}}_2 \left[\begin{array}{c} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{array} \right] \tag{66}$$

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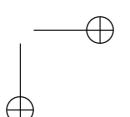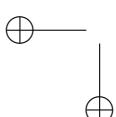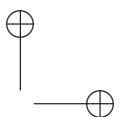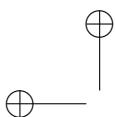
$$\underline{\underline{\mathrm{T}}}_1 = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix} \tag{67}$$

$$\underline{\underline{\mathrm{T}}}_2 = \begin{bmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & m^2 - n^2 \end{bmatrix} \tag{68}$$

$$m = \cos(\alpha) \qquad\qquad n = \sin(\alpha) \tag{69}$$

$$\underline{\underline{\mathrm{T}}}_1^{-1}(+\alpha) = \underline{\underline{\mathrm{T}}}_1(-\alpha) \qquad\qquad \underline{\underline{\mathrm{T}}}_2^{-1}(+\alpha) = \underline{\underline{\mathrm{T}}}_2(-\alpha) \tag{70}$$

$$\underline{\sigma} = \underline{\underline{\mathrm{D}}}\,\underline{\epsilon} \qquad\qquad \underline{\underline{\mathrm{D}}} \equiv \underline{\underline{\mathrm{D}}}_{xyz} = \underline{\underline{\mathrm{T}}}_1^{-1}\underline{\underline{\mathrm{D}}}_{123}\underline{\underline{\mathrm{T}}}_2 \tag{71}$$

$$\underline{\underline{\mathrm{G}}} = \begin{bmatrix} n^2 G_{z1} + m^2 G_{2z} & mnG_{z1} - mnG_{2z} \\ mnG_{z1} - mnG_{2z} & m^2 G_{z1} + n^2 G_{2z} \end{bmatrix}.$$

$$k_{\mathrm{x}}^* = \frac{12Fl}{Ebh^3} \tag{72}$$

$$m_{\mathrm{x}} = m_{\mathrm{x}}^* \qquad m_{\mathrm{y}} = 0 \qquad \kappa_{\mathrm{x}} = k_{\mathrm{x}}^* \qquad \kappa_{\mathrm{y}} = -\nu k_{\mathrm{x}}^*,$$

$$m_{\mathrm{x}} = m_{\mathrm{x}}^* \qquad m_{\mathrm{y}} = \nu m_{\mathrm{x}}^* \qquad \kappa_{\mathrm{x}} = \left(1 - \nu^2\right) k_{\mathrm{x}}^* \qquad \kappa_{\mathrm{y}} = 0.$$

$$g(y) \geq 0 \tag{73}$$
$$f(y) \geq 0 \tag{74}$$
$$g(y) \cdot f(y) = 0, \tag{75}$$

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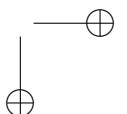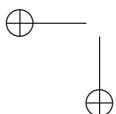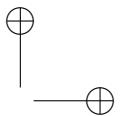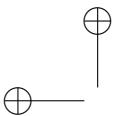
$$f(\xi, \eta) \stackrel{\text{def}}{=} \sum_i N_i(\xi, \eta) f_i \tag{76}$$

$$N_i(\xi, \eta) \stackrel{\text{def}}{=} \frac{1}{4} \left(1 \pm \xi\right)\left(1 \pm \eta\right), \tag{77}$$

$$\frac{\partial f}{\partial \xi} = \underbrace{\left(\frac{f_2 - f_1}{2}\right)}_{[\Delta f/\Delta \xi]_{12}} \underbrace{\left(\frac{1 - \eta}{2}\right)}_{N_1 + N_2} + \underbrace{\left(\frac{f_3 - f_4}{2}\right)}_{[\Delta f/\Delta \xi]_{43}} \underbrace{\left(\frac{1 + \eta}{2}\right)}_{N_4 + N_3} = a\eta + b \tag{78}$$

$$\frac{\partial f}{\partial \eta} = \left(\frac{f_4 - f_1}{2}\right)\left(\frac{1 - \xi}{2}\right) + \left(\frac{f_3 - f_2}{2}\right)\left(\frac{1 + \xi}{2}\right) = c\xi + d. \tag{79}$$

$$f(\xi, \eta) = \begin{bmatrix} N_1(\xi, \eta) & \cdots & N_i(\xi, \eta) & \cdots & N_n(\xi, \eta) \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_i \\ \vdots \\ f_n \end{bmatrix}$$

$$= \underline{\underline{N}}(\xi, \eta)\,\underline{f}, \tag{80}$$

$$\underline{x}\left(\underline{\xi}\right) = \underline{m}\left(\underline{\xi}\right) = \sum_{i=1}^{4} N_i\left(\underline{\xi}\right)\underline{x}_i, \tag{81}$$

$$\underline{m}\left(\underline{\xi}\right) = \begin{bmatrix} x(\xi, \eta) \\ y(\xi, \eta) \end{bmatrix}$$

$$x(\xi, \eta) = \sum_{i=1}^{4} N_i(\xi, \eta) x_i \qquad y(\xi, \eta) = \sum_{i=1}^{4} N_i(\xi, \eta) y_i.$$

$$f(\xi, \eta) \stackrel{\text{def}}{=} \sum_i N_i(\xi, \eta) f_i \tag{82}$$

$$\begin{bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}}_{\underline{\underline{J}}^{\top}(\xi, \eta; \underline{x}_i)} \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} \tag{83}$$

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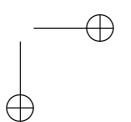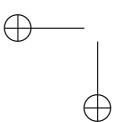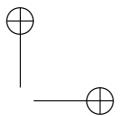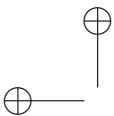
$$\begin{bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{bmatrix} = \sum_i \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} f_i. \tag{84}$$

$$\underline{\underline{J}}^\top(\xi, \eta) = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \tag{85}$$

$$= \sum_i \left(\begin{bmatrix} \frac{\partial N_i}{\partial \xi} & 0 \\ \frac{\partial N_i}{\partial \eta} & 0 \end{bmatrix} x_i + \begin{bmatrix} 0 & \frac{\partial N_i}{\partial \xi} \\ 0 & \frac{\partial N_i}{\partial \eta} \end{bmatrix} y_i \right) \tag{86}$$

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \left(\underline{\underline{J}}^\top \right)^{-1} \begin{bmatrix} \cdots & \frac{\partial N_i}{\partial \xi} & \cdots \\ \cdots & \frac{\partial N_i}{\partial \eta} & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ f_i \\ \vdots \end{bmatrix} \tag{87}$$

$$= \underbrace{\left(\underline{\underline{J}}^\top \right)^{-1} \begin{bmatrix} \frac{\partial \underline{N}}{\partial \xi} \\ \frac{\partial \underline{N}}{\partial \eta} \end{bmatrix}}_{\underline{\underline{L}}(\xi, \eta; \underline{x}_i), \text{ or just } \underline{\underline{L}}(\xi, \eta)} \underline{f} \tag{88}$$

$$\int_{-1}^{1} f(\xi) d\xi \approx \sum_{i=1}^{n} f(\xi_i) w_i; \tag{89}$$

$$p(\xi) \stackrel{\text{def}}{=} a_m \xi^m + a_{m-1} \xi^{m-1} + \ldots + a_1 \xi + a_0$$

$$\int_{-1}^{1} p(\xi) d\xi = \sum_{j=0}^{m} \frac{(-1)^j + 1}{j+1} a_j$$

$$r\left(a_j, (\xi_i, w_i)\right) \stackrel{\text{def}}{=} \sum_{i=1}^{n} p(\xi_i) w_i - \int_{-1}^{1} p(\xi) d\xi \tag{90}$$

$$\left\{ \frac{\partial r\left(a_j, (\xi_i, w_i)\right)}{\partial a_j} = 0, \quad j = 0 \ldots m \tag{91} \right.$$

$$\int_a^b g(x) dx = \int_{-1}^{1} g\left(m(\xi)\right) \frac{dm}{d\xi} d\xi \approx \sum_{i=1}^{n} g\left(m(\xi_i)\right) \frac{dm}{d\xi}\bigg|_{\xi=\xi_i} w_i. \tag{92}$$

$$m(x) = \underbrace{\left(\frac{1-\xi}{2} \right)}_{N_1} a + \underbrace{\left(\frac{1+\xi}{2} \right)}_{N_2} b.$$

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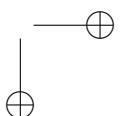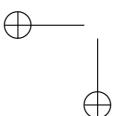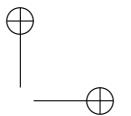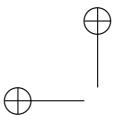
$$\frac{dm}{d\xi} = \frac{dN_1}{d\xi}a + \frac{dN_2}{d\xi}b = \frac{b-a}{2}$$

$$\int_a^b g(x)dx \approx \frac{b-a}{2}\sum_{i=1}^n g\left(\frac{b+a}{2} + \frac{b-a}{2}\xi_i\right)w_i. \tag{93}$$

$$\int_{-1}^1 \int_{-1}^1 f(\xi,\eta)\, d\xi d\eta \approx \sum_{i=1}^p \sum_{j=1}^q f(\xi_i,\eta_j)\, w_i w_j \tag{94}$$

$$\int_{-1}^1 \int_{-1}^1 f(\xi,\eta)\, d\xi d\eta \approx \sum_{l=1}^{pq} f\left(\underline{\xi}_l\right) w_l \tag{95}$$

$$\underline{\xi}_l = (\xi_i,\eta_j), \quad w_l = w_i w_j, \quad l = 1\ldots pq \tag{96}$$

$$dA_{xy} = \frac{1}{2!}\begin{vmatrix} 1 & x(\xi_P, \eta_P) & y(\xi_P, \eta_P) \\ 1 & x(\xi_P+d\xi, \eta_P) & y(\xi_P+d\xi, \eta_P) \\ 1 & x(\xi_P+d\xi, \eta_P+d\eta) & y(\xi_P+d\xi, \eta_P+d\eta) \end{vmatrix} +$$
$$+ \frac{1}{2!}\begin{vmatrix} 1 & x(\xi_P+d\xi, \eta_P+d\eta) & y(\xi_P+d\xi, \eta_P+d\eta) \\ 1 & x(\xi_P, \eta_P+d\eta) & y(\xi_P, \eta_P+d\eta) \\ 1 & x(\xi_P, \eta_P) & y(\xi_P, \eta_P) \end{vmatrix}. \tag{97}$$

$$\mathcal{A} = \frac{1}{2!}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}, \quad \mathcal{H} = \frac{1}{n!}\begin{vmatrix} 1 & \underline{x}_1 \\ 1 & \underline{x}_2 \\ \vdots & \vdots \\ 1 & \underline{x}_{n+1} \end{vmatrix} \tag{98}$$

$$dA_{xy} \approx \frac{1}{2!}\begin{vmatrix} 1 & x & y \\ 1 & x + x_{,\xi}d\xi & y + y_{,\xi}d\xi \\ 1 & x + x_{,\xi}d\xi + x_{,\eta}d\eta & y + y_{,\xi}d\xi + y_{,\eta}d\eta \end{vmatrix} +$$
$$+ \frac{1}{2!}\begin{vmatrix} 1 & x + x_{,\xi}d\xi + x_{,\eta}d\eta & y + y_{,\xi}d\xi + y_{,\eta}d\eta \\ 1 & x + x_{,\eta}d\eta & y + y_{,\eta}d\eta \\ 1 & x & y \end{vmatrix}$$

$$dA_{xy} = \frac{1}{2}\begin{vmatrix} 1 & x & y \\ 0 & x_{,\xi} & y_{,\xi} \\ 0 & x_{,\eta} & y_{,\eta} \end{vmatrix} d\xi d\eta + \frac{1}{2}\begin{vmatrix} 0 & x_{,\xi} & y_{,\xi} \\ 0 & x_{,\eta} & y_{,\eta} \\ 1 & x & y \end{vmatrix} d\xi d\eta$$
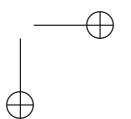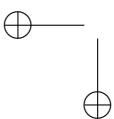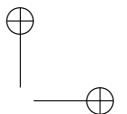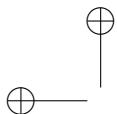
$$dA_{xy} = \frac{1}{2} \begin{vmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{vmatrix} d\xi d\eta + \frac{1}{2} \begin{vmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{vmatrix} d\xi d\eta$$

$$dA_{xy} = \underbrace{\begin{vmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{vmatrix}}_{\left| J^{\mathrm{T}}(\xi_P, \eta_P; \underline{x}, \underline{y}) \right|} dA_{\xi\eta} \tag{99}$$

$$\iint_{A_{xy}} g(x,y) dA_{xy} = \int_{-1}^{1} \int_{-1}^{1} g\left(x\left(\xi,\eta\right), y\left(\xi,\eta\right)\right) \left| J(\xi,\eta)\right| d\xi d\eta, \tag{100}$$

$$\iint_{A_{xy}} g(\underline{x}) dA_{xy} \approx \sum_{l=1}^{pq} g\left(\underline{x}\left(\underline{\xi}_l\right)\right) \left| J(\underline{\xi}_l)\right| w_l \tag{101}$$
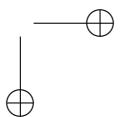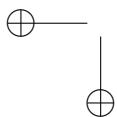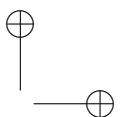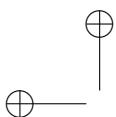
$$\begin{bmatrix} X(\xi,\eta) \\ Y(\xi,\eta) \\ Z(\xi,\eta) \end{bmatrix} = \sum_{i=1}^{n} N_i(\xi,\eta) \begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix}, \quad \begin{bmatrix} x(\xi,\eta) \\ y(\xi,\eta) \\ z(\xi,\eta) \end{bmatrix} = \sum_{i=1}^{n} N_i(\xi,\eta) \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}$$

$$(102)$$

$$\begin{bmatrix} u(\xi,\eta) \\ v(\xi,\eta) \\ w(\xi,\eta) \end{bmatrix} = \sum_{i=1}^{4} N_i(\xi,\eta) \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} \tag{103}$$

$$\begin{bmatrix} \theta(\xi,\eta) \\ \phi(\xi,\eta) \\ \psi(\xi,\eta) \end{bmatrix} = \sum_{i=1}^{4} N_i(\xi,\eta) \begin{bmatrix} \theta_i \\ \phi_i \\ \psi_i \end{bmatrix} \tag{104}$$

$$\underline{u} = \begin{bmatrix} \vdots \\ u_i \\ \vdots \end{bmatrix} \qquad \underline{v} = \begin{bmatrix} \vdots \\ v_i \\ \vdots \end{bmatrix} \qquad \underline{w} = \begin{bmatrix} \vdots \\ w_i \\ \vdots \end{bmatrix}$$

$$\underline{\theta} = \begin{bmatrix} \vdots \\ \theta_i \\ \vdots \end{bmatrix} \qquad \underline{\phi} = \begin{bmatrix} \vdots \\ \phi_i \\ \vdots \end{bmatrix} \qquad \underline{\psi} = \begin{bmatrix} \vdots \\ \psi_i \\ \vdots \end{bmatrix}$$

$$u(\xi,\eta) = \underline{\underline{N}}(\xi,\eta)\,\underline{u} \qquad\qquad v(\xi,\eta) = \underline{\underline{N}}(\xi,\eta)\,\underline{v}$$

$$\underline{d}^{\top} = \begin{bmatrix} \underline{u}^{\top} & \underline{v}^{\top} & \underline{w}^{\top} & \underline{\theta}^{\top} & \underline{\phi}^{\top} & \underline{\psi}^{\top} \end{bmatrix} \tag{105}$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \left(\underline{\underline{J}}'\right)^{-1} \underbrace{\begin{bmatrix} \cdots & \frac{\partial N_i}{\partial \xi} & \cdots \\ \cdots & \frac{\partial N_i}{\partial \eta} & \cdots \end{bmatrix}}_{\underline{\underline{L}}(\xi,\eta;\underline{x}_i) \text{ or just } \underline{\underline{L}}(\xi,\eta)} \begin{bmatrix} \vdots \\ u_i \\ \vdots \end{bmatrix} \tag{106}$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \underbrace{\begin{bmatrix} \underline{\underline{L}}(\xi,\eta) & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{L}}(\xi,\eta) \end{bmatrix}}_{\underline{\underline{Q}}(\xi,\eta)} \begin{bmatrix} \underline{u} \\ \underline{v} \end{bmatrix} \tag{107}$$

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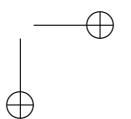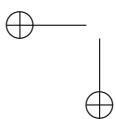
$$\begin{bmatrix} \frac{\partial \theta}{\partial x} \\ \frac{\partial \theta}{\partial y} \\ \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{bmatrix} = \underline{\underline{Q}}(\xi, \eta) \begin{bmatrix} \underline{\theta} \\ \underline{\phi} \end{bmatrix} \tag{108}$$

$$\begin{bmatrix} e_x \\ e_y \\ g_{xy} \end{bmatrix} = \underbrace{\begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & +1 & +1 & 0 \end{bmatrix}}_{\underline{\underline{H}}'} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \underline{\underline{H}}' \underline{\underline{Q}}(\xi, \eta) \begin{bmatrix} \underline{u} \\ \underline{v} \end{bmatrix} \tag{109}$$

$$\begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & +1 \end{bmatrix}}_{\underline{\underline{H}}''} \begin{bmatrix} \frac{\partial \theta}{\partial x} \\ \frac{\partial \theta}{\partial y} \\ \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{bmatrix} = \underline{\underline{H}}'' \underline{\underline{Q}}(\xi, \eta) \begin{bmatrix} \underline{\theta} \\ \underline{\phi} \end{bmatrix} \tag{110}$$

$$\underline{e} = \underbrace{\begin{bmatrix} \underline{\underline{H}}' \underline{\underline{Q}}(\xi, \eta) & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} \end{bmatrix}}_{\underline{\underline{B}}_e(\xi, \eta)} \underline{d} \tag{111}$$

$$\underline{\kappa} = \underbrace{\begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{H}}'' \underline{\underline{Q}}(\xi, \eta) & \underline{\underline{0}} \end{bmatrix}}_{\underline{\underline{B}}_\kappa(\xi, \eta)} \underline{d}. \tag{112}$$

$$\underline{\epsilon}(\xi, \eta, z) = \left(\underline{\underline{B}}_e(\xi, \eta) + z \underline{\underline{B}}_\kappa(\xi, \eta) \right) \underline{d}; \tag{113}$$

$$\begin{bmatrix} \bar{\gamma}_{zx} \\ \bar{\gamma}_{yz} \end{bmatrix} = \underline{\underline{L}}(\xi, \eta) \underline{w} + \begin{bmatrix} \underline{\underline{0}} & +\underline{\underline{N}}(\xi, \eta) \\ -\underline{\underline{N}}(\xi, \eta) & \underline{\underline{0}} \end{bmatrix} \begin{bmatrix} \underline{\theta} \\ \underline{\phi} \end{bmatrix}, \tag{114}$$

$$\begin{bmatrix} \bar{\gamma}_{zx} \\ \bar{\gamma}_{yz} \end{bmatrix} = \underbrace{\begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{L}}(\xi, \eta) & 0 & \underline{\underline{N}}(\xi, \eta) & \underline{\underline{0}} \\ & & & -\underline{\underline{N}}(\xi, \eta) & 0 & \end{bmatrix}}_{\underline{\underline{B}}_\gamma(\xi, \eta)} \underline{d} \tag{115}$$

$$\underline{d}^\top = \begin{bmatrix} \underline{u}^\top & \underline{v}^\top & \underline{w}^\top & \underline{\theta}^\top & \underline{\phi}^\top & \underline{\psi}^\top \end{bmatrix} \tag{116}$$

$$\underline{G}^\top = \begin{bmatrix} \underline{U}^\top & \underline{V}^\top & \underline{W}^\top & \underline{\Theta}^\top & \underline{\Phi}^\top & \underline{\Psi}^\top \end{bmatrix} \tag{117}$$

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$$\delta\,\Upsilon_{\mathrm{e}} = \delta\,\underline{\mathrm{d}}^{\top}\,\underline{\mathrm{G}}\,. \tag{118}$$

$$\underline{\sigma} = \underline{\underline{\mathrm{D}}}\,(z)\,\left(\underline{\underline{\mathrm{B}}}_{e}(\xi,\eta) + \underline{\underline{\mathrm{B}}}_{\kappa}(\xi,\eta)z\right)\,\underline{\mathrm{d}} \tag{119}$$

$$\delta\,\underline{\epsilon} = \left(\underline{\underline{\mathrm{B}}}_{e}(\xi,\eta) + \underline{\underline{\mathrm{B}}}_{\kappa}(\xi,\eta)z\right)\,\delta\,\underline{\mathrm{d}} \tag{120}$$

$$\underline{\mathrm{q}} = \left(\underline{\mathrm{a}}\,\underline{\underline{\mathrm{B}}}_{e}(\xi,\eta) + \underline{\mathrm{b}}\,\underline{\underline{\mathrm{B}}}_{\kappa}(\xi,\eta)\right)\,\underline{\mathrm{d}} \tag{121}$$

$$\underline{\mathrm{m}} = \left(\underline{\mathrm{b}}^{\top}\,\underline{\underline{\mathrm{B}}}_{e}(\xi,\eta) + \underline{\mathrm{c}}\,\underline{\underline{\mathrm{B}}}_{\kappa}(\xi,\eta)\right)\,\underline{\mathrm{d}}\,, \tag{122}$$

$$\delta\,\underline{\mathrm{e}} = \underline{\underline{\mathrm{B}}}_{e}(\xi,\eta)\,\delta\,\underline{\mathrm{d}} \tag{123}$$

$$\delta\,\underline{\kappa} = \underline{\underline{\mathrm{B}}}_{\kappa}(\xi,\eta)\,\delta\,\underline{\mathrm{d}}\,, \tag{124}$$

$$
\begin{aligned}
\delta\,\Upsilon_{\mathrm{i}}^{\dagger} &= \iint_{\mathcal{A}}\int_{h}\delta\,\underline{\epsilon}^{\top}\,\underline{\sigma}\,dz d\mathcal{A}\\
&= \iint_{\mathcal{A}}\int_{h}\left(\left(\underline{\underline{\mathrm{B}}}_{e} + \underline{\underline{\mathrm{B}}}_{\kappa}z\right)\,\delta\,\underline{\mathrm{d}}\right)^{\top}\,\underline{\underline{\mathrm{D}}}\,\left(\underline{\underline{\mathrm{B}}}_{e} + \underline{\underline{\mathrm{B}}}_{\kappa}z\right)\,\underline{\mathrm{d}}\,dz d\mathcal{A}\\
&= \delta\,\underline{\mathrm{d}}^{\top}\left[\iint_{\mathcal{A}}\int_{h}\left(\underline{\underline{\mathrm{B}}}_{e}^{\top} + \underline{\underline{\mathrm{B}}}_{\kappa}^{\top}z\right)\,\underline{\underline{\mathrm{D}}}\,\left(\underline{\underline{\mathrm{B}}}_{e} + \underline{\underline{\mathrm{B}}}_{\kappa}z\right)dz d\mathcal{A}\right]\,\underline{\mathrm{d}}\\
&= \delta\,\underline{\mathrm{d}}^{\top}\,\underline{\underline{\mathrm{K}}}^{\dagger}\,\underline{\mathrm{d}}\,, \tag{125}
\end{aligned}
$$

$$
\begin{aligned}
\delta\,\Upsilon_{\mathrm{i}}^{\dagger} &= \iint_{\mathcal{A}}\left(\delta\,\underline{\mathrm{e}}^{\top}\,\underline{\mathrm{q}} + \delta\,\underline{\kappa}^{\top}\,\underline{\mathrm{m}}\right)d\mathcal{A}\\
&= \delta\,\underline{\mathrm{d}}^{\top}\left[\iint_{\mathcal{A}}\left(\underline{\underline{\mathrm{B}}}_{e}^{\top}\left(\underline{\mathrm{a}}\,\underline{\underline{\mathrm{B}}}_{e} + \underline{\mathrm{b}}\,\underline{\underline{\mathrm{B}}}_{\kappa}\right) + \underline{\underline{\mathrm{B}}}_{\kappa}^{\top}\left(\underline{\mathrm{b}}\,\underline{\underline{\mathrm{B}}}_{e} + \underline{\mathrm{c}}\,\underline{\underline{\mathrm{B}}}_{\kappa}\right)\right)d\mathcal{A}\right]\,\underline{\mathrm{d}}\\
&= \delta\,\underline{\mathrm{d}}^{\top}\,\underline{\underline{\mathrm{K}}}_{\sigma}\,\underline{\mathrm{d}}\,, \tag{126}
\end{aligned}
$$

$$\left\{\underline{\mathrm{a}}\,,\,\underline{\mathrm{b}}\,,\,\underline{\mathrm{c}}\right\} = \int_{h}\underline{\underline{\mathrm{D}}}\,\left\{1, z, z^{2}\right\}dz,$$

$$\iiint_{\Omega}g(\xi,\eta,x,y,z)d\Omega = \tag{127}$$

$$= \int_{-1}^{+1}\int_{-1}^{+1}\int_{-\frac{h}{2}+o}^{+\frac{h}{2}+o}g(\xi,\eta,x(\xi,\eta),y(\xi,\eta),z)dz\,\left|\underline{\underline{\mathrm{J}}}\,(\xi,\eta)\right|\,d\xi d\eta,$$

$$\delta\,\Upsilon_i^\ddagger = \iint_{\mathcal{A}} \delta\,\underline{\gamma}_z^\top\,\underline{q}_z d\mathcal{A}$$

$$= \delta\,\underline{d}^\top \left[h \iint_{\mathcal{A}} \underline{\underline{B}}_\gamma^\top\,\underline{\underline{\Gamma}}\,\underline{\underline{B}}_\gamma d\mathcal{A} \right] \underline{d}$$

$$= \delta\,\underline{d}^\top\,\underline{\underline{K}}^\ddagger\,\underline{d}. \tag{128}$$

$$\delta\,\Upsilon_i = \delta\,\Upsilon_i^\dagger + \delta\,\Upsilon_i^\ddagger$$

$$= \delta\,\underline{d}^\top \left(\underline{\underline{K}}^\dagger + \underline{\underline{K}}^\ddagger \right) \underline{d}$$

$$= \delta\,\underline{d}^\top\,\underline{\underline{K}}\,\underline{d}. \tag{129}$$

$$\delta\,\underline{d}^\top\,\underline{G} = \delta\,\Upsilon_e = \delta\,\Upsilon_i = \delta\,\underline{d}^\top\,\underline{\underline{K}}\,\underline{d}, \quad \forall\,\delta\,\underline{d}, \tag{130}$$

$$\underline{G} = \underline{\underline{K}}\,\underline{d}; \tag{131}$$

a)

$+b$

$-a$ $+a$ x

undeformed domain

b)

$\frac{\alpha}{2}$ $\frac{\alpha}{2}$

$\frac{\alpha}{2}$

C_b $\epsilon_y > 0$

$\epsilon_y < 0$

exact solution,
pure in-plane bending

$\epsilon_x = -\frac{\alpha y}{2a}$

$\epsilon_y = \epsilon_z = \nu\frac{\alpha y}{2a}$

$\gamma_{xy} = 0$

$u = \frac{\alpha^2 E y^2}{8a^2}$

c)

$\frac{\alpha}{2}$ $\frac{\alpha}{2}$

C_iso4

four noded, isop. element,
in-plane trapezoidal mode

$\epsilon_x = -\frac{\alpha y}{2a}$

$\epsilon_y = 0,\ \epsilon_z = \frac{\nu}{1-\nu}\frac{\alpha y}{2a}$

$\gamma_{xy} = -\frac{\alpha x}{2a}$

$u = \left(1 + \frac{\nu^2}{1-\nu^2}\right)\frac{E\alpha^2 y^2}{8a^2} + \frac{G\alpha^2 x^2}{8a^2}$

d)

$\frac{\alpha}{2}$

residual (spurious)
shear deformation;

overall error assessment:

$\frac{C_\mathrm{iso4}}{C_\mathrm{b}} = \frac{1 + \frac{1-\nu}{2}\left(\frac{a}{b}\right)^2}{1-\nu^2}$

≈ 1.48 if $\nu = 0.3$, $\frac{a}{b} = 1$

$\gamma_{xy} = \frac{\alpha}{2}$ $\gamma_{xy} = 0$ $\gamma_{xy} = -\frac{\alpha}{2}$

24

(a)

(b)
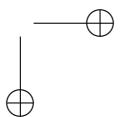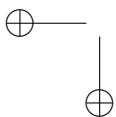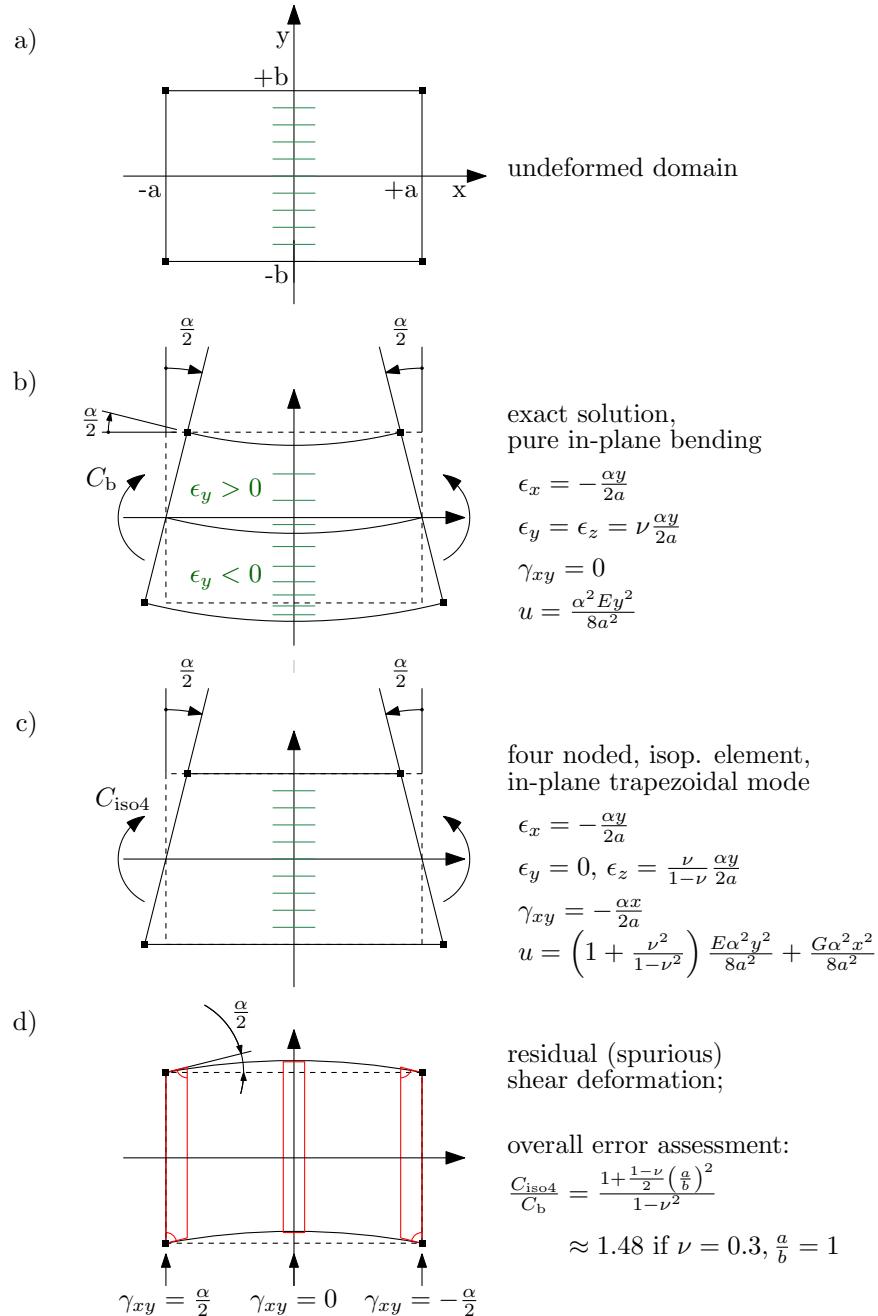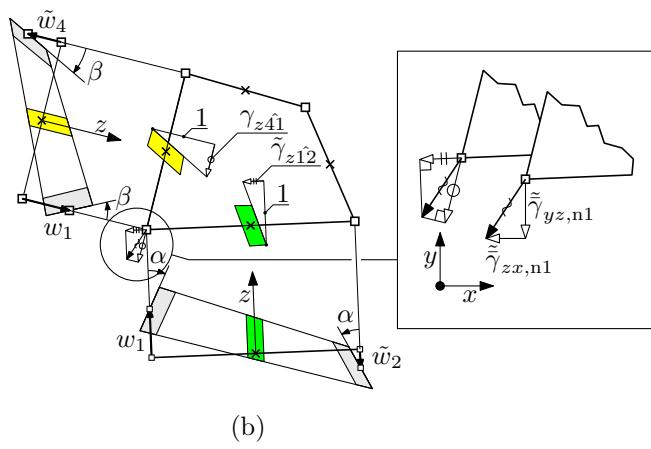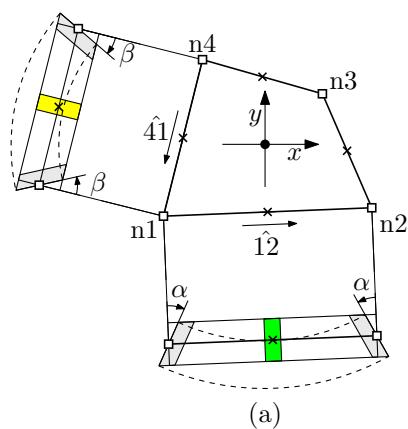
$$\underline{\underline{S}}\left(\xi, \eta, z\right) = \begin{bmatrix} \dots & \hat{u}_i(\xi, \eta, z) & \dots \\ \dots & \hat{v}_i(\xi, \eta, z) & \dots \\ \dots & \hat{w}_i(\xi, \eta, z) & \dots \end{bmatrix} \tag{132}$$

$$\underline{u}\left(\xi, \eta, z\right) = \underline{\underline{S}}\left(\xi, \eta, z\right)\underline{d}. \tag{133}$$

$$\underline{\dot{u}}\left(\xi, \eta, z\right) = \underline{\underline{S}}\left(\xi, \eta, z\right)\underline{\dot{d}} \tag{134}$$

$$E_{\mathrm{kin}} = \frac{1}{2}\iiint_\Omega \underline{\dot{u}}^{\top}\underline{\dot{u}}\,\rho d\Omega \tag{135}$$

$$E_{\mathrm{kin}} = \frac{1}{2}\iiint_\Omega \left[\underline{\underline{S}}\,\underline{\dot{d}}\right]^{\top}\left[\underline{\underline{S}}\,\underline{\dot{d}}\right]\rho d\Omega, \tag{136}$$

$$E_{\mathrm{kin}} = \frac{1}{2}\underline{\dot{d}}^{\top}\left[\iiint_\Omega \underline{\underline{S}}^{\top}\underline{\underline{S}}\,\rho d\Omega\right]\underline{\dot{d}} = \frac{1}{2}\underline{\dot{d}}^{\top}\underline{\underline{M}}\,\underline{\dot{d}}. \tag{137}$$

$$\underline{\underline{M}} = \iiint_\Omega \underline{\underline{S}}^{\top}\underline{\underline{S}}\,\rho d\Omega, \tag{138}$$

$$\begin{aligned}
\underline{\dot{d}}^{\top}\underline{G} = \frac{dE_{\mathrm{kin}}}{dt} &= \frac{d}{dt}\left(\frac{1}{2}\underline{\dot{d}}^{\top}\underline{\underline{M}}\,\underline{\dot{d}}\right) \\
&= \frac{1}{2}\left(\underline{\ddot{d}}^{\top}\underline{\underline{M}}\,\underline{\dot{d}} + \underline{\dot{d}}^{\top}\underline{\underline{M}}\,\underline{\ddot{d}}\right) \\
&= \underline{\dot{d}}^{\top}\underline{\underline{M}}\,\underline{\ddot{d}}.
\end{aligned}$$

$$\underline{G} = \underline{\underline{M}}\,\underline{\ddot{d}} \tag{139}$$

$$\delta\underline{u}\left(\xi, \eta, z\right) = \underline{\underline{S}}\left(\xi, \eta, z\right)\delta\underline{d}, \tag{140}$$

$$\begin{aligned}
\delta\underline{d}^{\top}\underline{F} &= \iiint_\Omega \left(\delta\underline{u}\right)^{\top}\underline{p}\,d\Omega \\
&= \iiint_\Omega \left(\underline{\underline{S}}\,\delta\underline{d}\right)^{\top}\underline{p}\,d\Omega \\
&= \delta\underline{d}^{\top}\iiint_\Omega \underline{\underline{S}}^{\top}\underline{p}\,d\Omega,
\end{aligned}$$

$$\underline{F} = \iiint_\Omega \underline{\underline{S}}^{\top}\underline{p}\,d\Omega \tag{141}$$
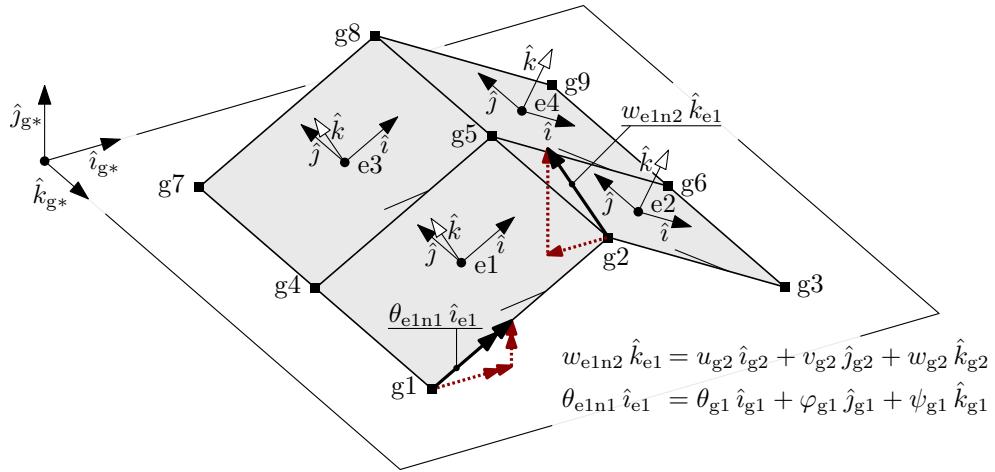
$$\underline{G}_{ej} = \underline{\underline{K}}_{ej}\,\underline{d}_{ej} \tag{142}$$

$$\underline{d}_{gl} = \begin{bmatrix} u_{gl} \\ v_{gl} \\ w_{gl} \\ \theta_{gl} \\ \varphi_{gl} \\ \psi_{gl} \end{bmatrix}. \tag{143}$$

$$\underline{d}_g^\top = \begin{bmatrix} \underline{d}_{g1}^\top & \underline{d}_{g2}^\top & \cdots & \underline{d}_{gl}^\top & \cdots & \underline{d}_{gn}^\top \end{bmatrix} \tag{144}$$

$$\underline{F}_g^\top = \begin{bmatrix} \underline{F}_{g1}^\top & \underline{F}_{g2}^\top & \cdots & \underline{F}_{gl}^\top & \cdots & \underline{F}_{gn}^\top \end{bmatrix}; \tag{145}$$

$$\underline{R}_g^\top = \begin{bmatrix} \underline{R}_{g1}^\top & \underline{R}_{g2}^\top & \cdots & \underline{R}_{gl}^\top & \cdots & \underline{R}_{gn}^\top \end{bmatrix} \tag{146}$$

$$w_{e1n2} = \langle \hat{k}_{e1}, \hat{i}_{g2}\rangle u_{g2} + \langle \hat{k}_{e1}, \hat{j}_{g2}\rangle v_{g2} + \langle \hat{k}_{e1}, \hat{k}_{g2}\rangle w_{g2} \tag{147}$$

$$\theta_{e1n1} = \langle \hat{i}_{e1}, \hat{i}_{g1}\rangle \theta_{g1} + \langle \hat{i}_{e1}, \hat{j}_{g1}\rangle \phi_{g1} + \langle \hat{i}_{e1}, \hat{k}_{g1}\rangle \psi_{g1} \tag{148}$$

$$\begin{aligned}
\left[\underline{\underline{P}}_{e1}\right]_{10,7} &= \langle \hat{k}_{e1}, \hat{i}_{g2}\rangle & \left[\underline{\underline{P}}_{e1}\right]_{13,4} &= \langle \hat{i}_{e1}, \hat{i}_{g1}\rangle \\
\left[\underline{\underline{P}}_{e1}\right]_{10,8} &= \langle \hat{k}_{e1}, \hat{j}_{g2}\rangle & \left[\underline{\underline{P}}_{e1}\right]_{13,5} &= \langle \hat{i}_{e1}, \hat{j}_{g1}\rangle \\
\left[\underline{\underline{P}}_{e1}\right]_{10,9} &= \langle \hat{k}_{e1}, \hat{k}_{g2}\rangle & \left[\underline{\underline{P}}_{e1}\right]_{13,6} &= \langle \hat{i}_{e1}, \hat{k}_{g1}\rangle,
\end{aligned}$$

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node	X	Y	Z
g1	$-ac$	0	$+a$
g2	0	$+as$	$+a$
g3	$+ac$	0	$+a$
g4	$-ac$	0	0
g5	0	$+as$	0
g6	$+ac$	0	0
g7	$-ac$	0	$-a$
g8	0	$+as$	$-a$
g9	$+ac$	0	$-a$

	$u_{\mathrm{n}i}$	$v_{\mathrm{n}i}$	$w_{\mathrm{n}i}$	$\theta_{\mathrm{n}i}$	$\varphi_{\mathrm{n}i}$	$\psi_{\mathrm{n}i}$
$U_{\mathrm{n}i}$						
$V_{\mathrm{n}i}$						
$W_{\mathrm{n}i}$						
$\Theta_{\mathrm{n}i}$						
$\Phi_{\mathrm{n}i}$						
$\Psi_{\mathrm{n}i}$						

$i = 1 \ldots 4$

	n1	n2	n3	n4
e1	g1	g2	g5	g4
e2	g2	g3	g6	g5
e3	g4	g5	g8	g7
e4	g5	g6	g9	g8

$\underline{\underline{P}}_{e1}$, columns: $\underline{d}_{g1}$, $\underline{d}_{g2}$, $\underline{d}_{g3}$, $\underline{d}_{g4}$, $\underline{d}_{g5}$, $\underline{d}_{g6}$, $\underline{d}_{g7}$, $\underline{d}_{g8}$, $\underline{d}_{g9}$; rows: u_{e1ni}, v_{e1ni}, w_{e1ni}, θ_{e1ni}, φ_{e1ni}, ψ_{e1ni}

$\underline{\underline{P}}_{e2}$, columns: $\underline{d}_{g1}$, $\underline{d}_{g2}$, $\underline{d}_{g3}$, $\underline{d}_{g4}$, $\underline{d}_{g5}$, $\underline{d}_{g6}$, $\underline{d}_{g7}$, $\underline{d}_{g8}$, $\underline{d}_{g9}$; rows: u_{e2ni}, v_{e2ni}, w_{e2ni}, θ_{e2ni}, φ_{e2ni}, ψ_{e2ni}

$\underline{\underline{P}}_{e3}$, columns: $\underline{d}_{g1}$, $\underline{d}_{g2}$, $\underline{d}_{g3}$, $\underline{d}_{g4}$, $\underline{d}_{g5}$, $\underline{d}_{g6}$, $\underline{d}_{g7}$, $\underline{d}_{g8}$, $\underline{d}_{g9}$; rows: u_{e3ni}, v_{e3ni}, w_{e3ni}, θ_{e3ni}, φ_{e3ni}, ψ_{e3ni}

$\underline{\underline{P}}_{e4}$, columns: $\underline{d}_{g1}$, $\underline{d}_{g2}$, $\underline{d}_{g3}$, $\underline{d}_{g4}$, $\underline{d}_{g5}$, $\underline{d}_{g6}$, $\underline{d}_{g7}$, $\underline{d}_{g8}$, $\underline{d}_{g9}$; rows: u_{e4ni}, v_{e4ni}, w_{e4ni}, θ_{e4ni}, φ_{e4ni}, ψ_{e4ni}

(a)

(b)

(c)

(d)

$$\underline{d}_{ej} = \underline{\underline{P}}_{ej}\, \underline{d}_{g}, \quad \forall j. \tag{149}$$

$$\underline{G}_{ej} = \underline{\underline{K}}_{ej}\, \underline{\underline{P}}_{ej}\, \underline{d}_{g}, \quad \forall j; \tag{150}$$

$$\delta\,\underline{d}_{g}^{\top}\, \underline{G}_{g\leftarrow ej} = \left(\underline{\underline{P}}_{ej}\, \delta\,\underline{d}_{g}\right)^{\top} \underline{G}_{ej}, \quad \forall\ \delta\,\underline{d}_{g} \tag{151}$$

$$\underline{G}_{g\leftarrow ej} = \underline{\underline{P}}_{ej}^{\top}\, \underline{G}_{ej} \tag{152}$$

$$\underline{G}_{g\leftarrow ej} = \underline{\underline{P}}_{ej}^{\top}\, \underline{\underline{K}}_{ej}\, \underline{\underline{P}}_{ej}\, \underline{d}_{g}; \tag{153}$$

$$\underline{G}_{g} = \sum_{j} \underline{G}_{g\leftarrow ej} = \left(\sum_{j} \underbrace{\underline{\underline{P}}_{ej}^{\top}\, \underline{\underline{K}}_{ej}\, \underline{\underline{P}}_{ej}}_{\underline{\underline{K}}_{g\leftarrow ej}} \right) \underline{d}_{g} = \underline{\underline{K}}_{g}\, \underline{d}_{g}, \tag{154}$$
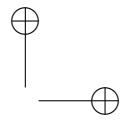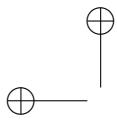
$$b_{\mathrm{e}j} = (i_{\max} - i_{\min} + 1)\, l, \tag{155}$$

$$b = \max_{\mathrm{e}j} b_{\mathrm{e}j} \tag{156}$$

$$\underline{\mathbf{F}}_{\mathrm{g}} = \sum_{j} \underline{\underline{\mathbf{P}}}_{\mathrm{e}j}^{\top}\, \underline{\mathbf{F}}_{\mathrm{e}j}; \tag{157}$$

(a)

(b)

I: $3\,d_1 - 0\,d_2 + 1\,d_3 = 0$
II: $0\,d_1 + 1\,d_2 + 0\,d_3 = 0.2$

(a) configuration space

(b) reaction space

$$\sum_i \alpha_{ji} d_i = \underline{\alpha}_j^\top \, \underline{d} = \beta_j$$

$$\underline{\alpha}_{\mathrm{I}}^\top = \begin{bmatrix} 3 & 0 & 1 \end{bmatrix} \qquad\qquad \beta_{\mathrm{I}} = 0$$
$$\underline{\alpha}_{\mathrm{II}}^\top = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \qquad\qquad \beta_{\mathrm{II}} = 0.2$$
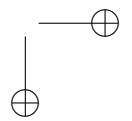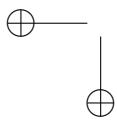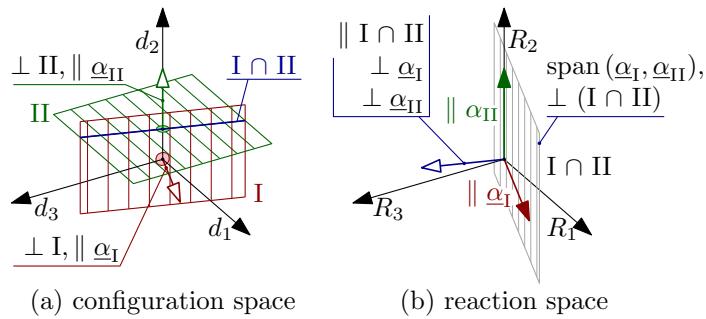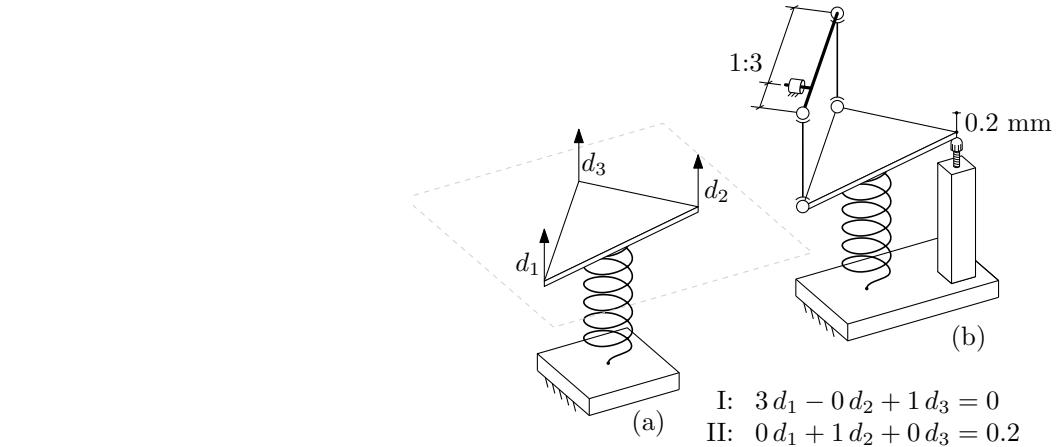
$$\sum_i \alpha_{ji} d_i = \underline{\alpha}_j^\top \, \underline{d} = \beta_j, \quad j = 1 \ldots m \tag{158}$$

$$\underline{\underline{\mathcal{L}}}^\top \underline{d} = \underline{\beta}\,. \tag{159}$$

$$\underline{\underline{\mathcal{L}}}^\top \delta\underline{d} = \underline{0}\,, \tag{160}$$

$$\underline{R} = -\underline{\underline{\mathcal{L}}}\,\underline{\ell}\,, \tag{161}$$
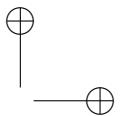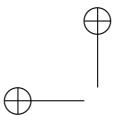
$$\underline{\mathrm{R}}^{j} = -\begin{bmatrix} \vdots \\ \alpha_{ji} \\ \vdots \end{bmatrix} \ell_{j} \tag{162}$$

$$\underline{\underline{\mathrm{K}}}\,\underline{\mathrm{d}} = \underline{\mathrm{F}} + \underline{\mathrm{R}}. \tag{163}$$

$$\underline{\underline{\mathrm{K}}}\,\underline{\mathrm{d}} + \underline{\underline{\mathcal{L}}}\,\underline{\ell} = \underline{\mathrm{F}}$$

$$\begin{bmatrix} \underline{\underline{\mathrm{K}}} & \underline{\underline{\mathcal{L}}} \\ \underline{\underline{\mathcal{L}}}^{\top} & \underline{\underline{0}} \end{bmatrix} \begin{bmatrix} \underline{\mathrm{d}} \\ \underline{\ell} \end{bmatrix} = \begin{bmatrix} \underline{\mathrm{F}} \\ \underline{\beta} \end{bmatrix}, \tag{164}$$

$$\frac{1}{2}\underline{\mathrm{d}}^{\top}\underline{\underline{\mathrm{K}}}\,\underline{\mathrm{d}} - \underline{\mathrm{d}}^{\top}\underline{\mathrm{F}} + \underline{\ell}^{\top}\left(\underline{\underline{\mathcal{L}}}^{\top}\underline{\mathrm{d}} - \underline{\beta}\right), \tag{165}$$

$$\frac{1}{2}\underline{\mathrm{d}}^{\top}\underline{\underline{\mathrm{K}}}\,\underline{\mathrm{d}} - \underline{\mathrm{d}}^{\top}\underline{\mathrm{F}}$$

$$\underline{\underline{\mathcal{L}}}^{\top}\underline{\mathrm{d}} - \underline{\beta} = \underline{0}$$

$$\underline{\mathrm{R}} = -\underline{\underline{\mathcal{L}}}\,\underline{\ell}^{*}.$$

$$d_{k} = \sum_{i \neq k}\left(-\frac{\alpha_{ji}}{\alpha_{jk}}\right)d_{i} + \left(\frac{\beta_{j}}{\alpha_{jk}}\right), \quad j = 1\ldots m \tag{166}$$

$$\begin{bmatrix} \underline{\mathrm{r}} \\ \underline{\mathrm{t}} \end{bmatrix} = \underbrace{\begin{bmatrix} \underline{\underline{\mathrm{E}}}_{\mathrm{R}} \\ \underline{\underline{\mathrm{E}}}_{\mathrm{T}} \end{bmatrix}}_{\underline{\underline{\mathrm{E}}}}\,\underline{\mathrm{d}} \qquad\qquad \underline{\mathrm{d}} = \underbrace{\begin{bmatrix} \underline{\underline{\mathrm{E}}}_{\mathrm{R}}^{\top} & \underline{\underline{\mathrm{E}}}_{\mathrm{T}}^{\top} \end{bmatrix}}_{\underline{\underline{\mathrm{E}}}^{\top} \equiv \underline{\underline{\mathrm{E}}}^{-1}}\begin{bmatrix} \underline{\mathrm{r}} \\ \underline{\mathrm{t}} \end{bmatrix},$$

$$\underline{\mathrm{r}} = \underline{\underline{\mathrm{E}}}_{\mathrm{R}}\,\underline{\mathrm{d}} \qquad\qquad \underline{\mathrm{t}} = \underline{\underline{\mathrm{E}}}_{\mathrm{T}}\,\underline{\mathrm{d}},$$

$$t_{j} = \sum_{h=1}^{n-m}\lambda_{jh}r_{h} + \delta_{j}, \quad j = 1\ldots m \tag{167}$$

$$\underline{\mathrm{t}} = \underline{\underline{\lambda}}\,\underline{\mathrm{r}} + \underline{\delta}, \tag{168}$$

$$\underline{\mathrm{d}} = \left(\underline{\underline{\mathrm{E}}}^{\top}\begin{bmatrix} \underline{\underline{\mathrm{I}}} \\ \underline{\underline{\lambda}} \end{bmatrix}\right)\underline{\mathrm{r}} + \left(\underline{\underline{\mathrm{E}}}^{\top}\begin{bmatrix} \underline{0} \\ \underline{\delta} \end{bmatrix}\right)$$

$$= \underline{\underline{\Lambda}}\,\underline{\mathrm{r}} + \underline{\Delta}; \tag{169}$$

h

retained dof.	d_i	i	$\cdots 1 \cdots 0 \cdots$			0
$d_i = r_h$	$\cdot$			$\cdot$		$\cdot$
tied dof.	d_k	k	$\cdots \lambda_{kh}=0 \cdots$	r_h		$\Delta_k = a$
fix.disp. kind $d_k = a$	$\cdot$ $=$			$\cdot$ $+$		$\cdot$
tied dof.	d_k	k	$\cdots \lambda_{kh} \cdots$	$\cdot$		Δ_k
general	$\cdot$			$\cdot$		$\cdot$

$$\underline{d} \quad = \quad \underline{\underline{\Lambda}} \qquad \underline{r} \quad + \quad \underline{\Delta}$$

$$\delta \underline{d} = \underline{\underline{\Lambda}}\, \delta \underline{r} = \underline{\Lambda}_1\, \delta r_1 + \underline{\Lambda}_2\, \delta r_2 + \ldots + \underline{\Lambda}_{n-m}\, \delta r_{n-m} \tag{170}$$

$$\langle \underline{\Lambda}_h, \underline{R} \rangle = 0 \quad h = 1 \ldots n - m, \tag{171}$$

or, equivalently,

$$\underline{\underline{\Lambda}}^{\top} \underline{R} = \underline{0}. \tag{172}$$

$$\underline{\underline{K}} \left(\underline{\underline{\Lambda}}\, \underline{r} + \underline{\Delta} \right) = \underline{F} + \underline{R} \tag{173}$$

$$\underline{\underline{K}}\, \underline{\underline{\Lambda}}\, \underline{r} = \left(\underline{F} - \underline{\underline{K}}\, \underline{\Delta} \right) + \underline{R}, \tag{174}$$

$$\underbrace{\underline{\underline{\Lambda}}^{\top} \underline{\underline{K}}\, \underline{\underline{\Lambda}}}_{\underline{\underline{K}}_{R}}\, \underline{r} = \underbrace{\underline{\underline{\Lambda}}^{\top} \left(\underline{F} - \underline{\underline{K}}\, \underline{\Delta} \right)}_{\underline{F}_{R}} + \underbrace{\underline{\underline{\Lambda}}^{\top} \underline{R}}_{=0}, \tag{175}$$

$$\underline{\underline{K}}_{R}\, \underline{r} = \underline{F}_{R} \tag{176}$$

$$\underline{d}^{*} = \underline{\underline{\Lambda}}\, \underline{r}^{*} + \underline{\Delta}; \tag{177}$$

$$\underline{R}^{*} = \underline{\underline{K}} \left(\underline{\underline{\Lambda}}\, \underline{r}^{*} + \underline{\Delta} \right) - \underline{F}. \tag{178}$$

$$\underline{d}_{ej}^{*} = \underline{\underline{P}}_{ej}\, \underline{d}^{*}. \tag{179}$$

$$\underline{e} = \underline{\underline{B}}_{ej}^{e}(\xi,\eta)\, \underline{d}_{ej}^{*} \qquad\qquad \underline{\kappa} = \underline{\underline{B}}_{ej}^{\kappa}(\xi,\eta)\, \underline{d}_{ej}^{*} \tag{180}$$

$$\underline{\epsilon} = \left(\underline{\underline{B}}_{ej}^{e}(\xi,\eta) + \underline{\underline{B}}_{ej}^{\kappa}(\xi,\eta) z \right) \underline{d}_{ej}^{*}. \tag{181}$$

$$\underline{\gamma}_{z} = \underline{\underline{B}}_{ej}^{\gamma}(\xi,\eta)\, \underline{d}_{ej}^{*}. \tag{182}$$
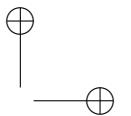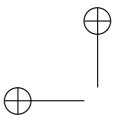
$$
\begin{bmatrix} u_i \\ v_i \\ w_i \\ \theta_i \\ \phi_i \\ \psi_i \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & +(z_i - z_C) & -(y_i - y_C) \\ 0 & 1 & 0 & -(z_i - z_C) & 0 & +(x_i - x_C) \\ 0 & 0 & 1 & +(y_i - y_C) & -(x_i - x_C) & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\underline{\underline{\mathbb{L}}}_i} \cdot \begin{bmatrix} u_C \\ v_C \\ w_C \\ \theta_C \\ \phi_C \\ \psi_C \end{bmatrix}
$$

$$(183)$$

$$
\begin{bmatrix} U_i \\ V_i \\ W_i \end{bmatrix} = q_i \left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} 0 & d & -f \\ -d & 0 & e \\ f & -e & 0 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} \right), \qquad (184)
$$

$$
\begin{bmatrix} U_C \\ V_C \\ W_C \end{bmatrix} = \sum_i \begin{bmatrix} U_i \\ V_i \\ W_i \end{bmatrix}, \qquad \begin{bmatrix} \Theta_C \\ \Phi_C \\ \Psi_C \end{bmatrix} = \sum_i \begin{bmatrix} U_i \\ V_i \\ W_i \end{bmatrix} \wedge \begin{bmatrix} x_i - x_C \\ y_i - y_C \\ z_i - z_C \end{bmatrix} \qquad (185)
$$
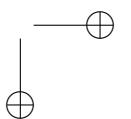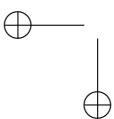
$$\underline{\underline{K}}\,\underline{d} = \underline{F}\,, \tag{186}$$

$$\ddot{\underline{d}} = \underbrace{\begin{bmatrix} \cdots & \underline{t}_l & \cdots \end{bmatrix}}_{\underline{\underline{T}}} \underbrace{\begin{bmatrix} \vdots \\ \alpha_l \\ \vdots \end{bmatrix}}_{\underline{\alpha}}, \tag{187}$$

$$\underline{\underline{T}}^{\top}\,\underline{\underline{M}}\,\underline{\underline{T}}\,\underline{\alpha} = \underline{\underline{T}}^{\top}\,\underline{F} \tag{188}$$

$$\underline{\underline{M}}\,\underline{\underline{T}}\,\underline{\alpha} = \underline{F}\,[+\underline{R}_l]$$

$$\underline{\underline{K}}\,\underline{d} = \underline{F} - \underline{\underline{M}}\,\underline{\underline{T}}\,\underline{\alpha}\,, \tag{189}$$
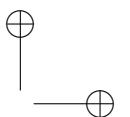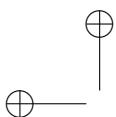
$$\underline{\underline{M}}\,\ddot{\underline{d}} + \underline{\underline{C}}\,\dot{\underline{d}} + \underline{\underline{K}}\,\underline{d} = \underline{f}\,(t), \quad \underline{d} = \underline{d}\,(t) \tag{190}$$

$$\underline{f}\,(t) = \frac{\bar{\underline{f}}\,e^{j\omega t} + \bar{\underline{f}}^{*}e^{-j\omega t}}{2} = \mathrm{Re}(\bar{\underline{f}}\,e^{j\omega t}) \tag{191}$$

$$\underline{f}\,(t) = \bar{\underline{f}}\,e^{j\omega t} \tag{192}$$

$$\mathrm{Re}(\bar{\underline{f}}\,e^{j\omega t}) = \mathrm{Re}(\bar{\underline{f}})\cos\omega t - \mathrm{Im}(\bar{\underline{f}})\sin\omega t \tag{193}$$

$$\underline{d}\,(t) = \bar{\underline{d}}\,e^{j\omega t} \tag{194}$$

$$\left(-\omega^2\,\underline{\underline{M}} + j\omega\,\underline{\underline{C}} + \underline{\underline{K}}\right)\bar{\underline{d}} = \bar{\underline{f}} \tag{195}$$

$$\left(-\omega^2\,\underline{\underline{M}} + \underline{\underline{K}}\right)\bar{\underline{d}} = \underline{0} \tag{196}$$

$$\left(\underline{\underline{M}}^{-1}\,\underline{\underline{K}} - \omega^2\,\underline{\underline{I}}\right)\hat{\underline{d}} = \underline{0}; \tag{197}$$

$$m_i = \hat{\underline{d}}_i^{\top}\,\underline{\underline{M}}\,\hat{\underline{d}}_i = 1 \tag{198}$$

$$\underline{x}\,(t) = a\,\hat{\underline{d}}_i\sin(\omega_i t) \tag{199}$$

$$f(t) = \hat{\underline{f}}\,\cos(\omega_i t), \tag{200}$$

$$\underbrace{\left(-\omega_i^2\,\underline{\underline{M}} + \underline{\underline{K}}\right)\hat{\underline{d}}_i}_{=\,\underline{0}}\,a_i\sin(\omega_i t) + \omega_i a_i\,\underline{\underline{C}}\,\hat{\underline{d}}_i\cos(\omega_i t) = \bar{\underline{f}}\cos(\omega_i t). \tag{201}$$

$$a_i = \frac{\hat{\underline{d}}^{\top}\bar{\underline{f}}}{\omega_i\,\hat{\underline{d}}^{\top}\,\underline{\underline{C}}\,\hat{\underline{d}}_i} \tag{202}$$

$$\hat{\underline{d}}_j^{\top}\,\underline{\underline{M}}\,\hat{\underline{d}}_i = m_i\delta_{ij} \qquad\qquad \hat{\underline{d}}_j^{\top}\,\underline{\underline{K}}\,\hat{\underline{d}}_i = m_i\omega_i^2\delta_{ij} \tag{203}$$

$$\underline{\underline{\Xi}} = \begin{bmatrix} \hat{\underline{d}}_1 & \cdots & \hat{\underline{d}}_l & \cdots & \hat{\underline{d}}_m \end{bmatrix}, \tag{204}$$

$$\bar{\underline{d}} = \underline{\underline{\Xi}}\,\bar{\xi} \tag{205}$$

.

$$\underline{\underline{\Xi}}^{\top}\,\underline{\underline{M}}\,\underline{\underline{\Xi}} = \underline{\underline{I}} \qquad\qquad \underline{\underline{\Xi}}^{\top}\,\underline{\underline{K}}\,\underline{\underline{\Xi}} = \underline{\underline{\Omega}} = \mathrm{diag}(\omega_l^2); \tag{206}$$

$$\underline{\underline{C}} = \alpha\,\underline{\underline{M}} + \beta\,\underline{\underline{K}} \tag{207}$$

$$\underline{\underline{\Xi}}^{\top}\left(-\omega^2\,\underline{\underline{M}} + j\omega\,\underline{\underline{C}} + \underline{\underline{K}}\right)\underline{\underline{\Xi}}\,\bar{\xi} = \underline{\underline{\Xi}}^{\top}\bar{\underline{f}} \tag{208}$$

$$\left(-\omega^2\,\underline{\underline{I}} + j\omega\left(\alpha\,\underline{\underline{I}} + \beta\,\underline{\underline{\Omega}}\right) + \underline{\underline{\Omega}}\right)\bar{\xi} = \underline{\underline{\Xi}}^{\top}\bar{\underline{f}}, \tag{209}$$

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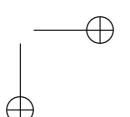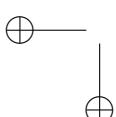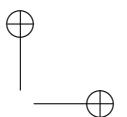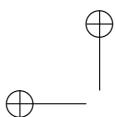
$$\left(-\omega^2 + j\omega\left(\alpha + \beta\omega_l^2\right) + \omega_l^2\right)\xi_l = q_l, \quad j = 1\ldots m \qquad (210)$$

$$\xi_l(t) = \text{Re}(\bar{\xi}_l)\cos\omega t - \text{Im}(\bar{\xi}_l)\sin\omega t$$
$$= \left|\bar{\xi}_l\right|\cos\left(\omega t + \psi_l - \phi_l\right)$$

$$a_l = 1 - r_l^2 \qquad\qquad b_l = 2\zeta_l r_l \qquad\qquad r_l = \frac{\omega}{\omega_l}$$

$$\left|\bar{\xi}_l\right| = \frac{|\bar{q}_l|}{\omega_l^2}\frac{1}{\sqrt{a_l^2 + b_l^2}}$$
$$\psi_l = \arg(\bar{q}_l)$$
$$\phi_l = \arg(a_l + jb_l)$$

$$\text{Re}(\bar{\xi}_l) = \frac{1}{\omega_l^2}\frac{a_l\,\text{Re}(\bar{q}_l) + b_l\,\text{Im}(\bar{q}_l)}{a_l^2 + b_l^2}$$
$$\text{Im}(\bar{\xi}_l) = \frac{1}{\omega_l^2}\frac{a_l\,\text{Im}(\bar{q}_l) - b_l\,\text{Re}(\bar{q}_l)}{a_l^2 + b_l^2}.$$

$$N_{AB} = N_{CA} = +\frac{F}{\sqrt{3}}$$
$$N_{BC} = -\frac{F}{2\sqrt{3}}$$
$$N_{\text{crit},i} = -\frac{\pi^2 i^2 EJ}{l^2}$$

$$\lambda_1 = -\frac{\pi^2 EJ}{l^2} \frac{\sqrt{3}}{F}$$

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minimum λ_i in absolute value

$$\lambda_2 = +\frac{\pi^2 EJ}{l^2} \frac{2\sqrt{3}}{F}$$

minimum $\lambda_i > 0$
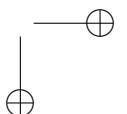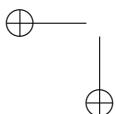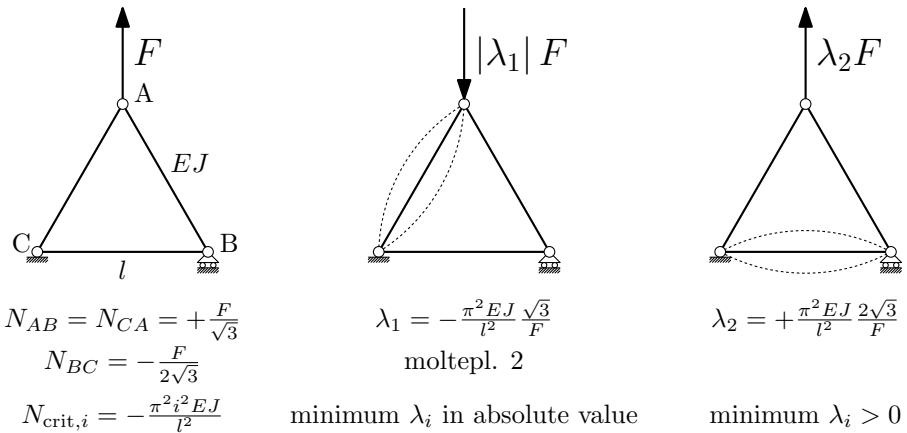
$$\delta U_{\text{i}} = \iiint_V \delta\, \underline{\epsilon}^\top \left(\underline{\sigma}_0 + \underline{\underline{D}}\, \underline{\epsilon} \right) dV$$
$$= \iiint_V \left[\underline{\underline{B}}(\underline{d}) \delta\underline{d} \right]^\top \left(\underline{\sigma}_0 + \underline{\underline{D}}\, \underline{\underline{B}}(\underline{d})\, \underline{d} \right) dV$$
$$= \dots$$
$$= \delta\underline{d} \left(\left(\underline{\underline{K}}_{\text{ej}}^{\text{M}} + \underline{\underline{K}}_{\text{ej}}^{\text{G}} \right) \underline{d} + \text{o}(\underline{d}) \right).$$

$$\left(\underline{\underline{K}}^{\text{M}} + \lambda \underline{\underline{K}}^{\text{G}} \right) \delta\underline{d} = \delta\underline{F} \tag{211}$$

$$\left(\underline{\underline{K}}^{\text{M}} + \lambda_i \underline{\underline{K}}^{\text{G}} \right) \delta\hat{\underline{d}}_i = 0 \tag{212}$$

Bibliography

[1] A. E. H. Love, *A treatise on the mathematical theory of elasticity.* Cambridge university press, 2013.